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INTEGER TURBINE BALANCING PROBLEM (*)

The integer turbine balancing problem $P(n)$ requires: Place n points with integer weights $1, 2, 3, \dots, n$ equidistantly on the circle in such a way that their centre of mass is as close to the centre of the circle as possible. In this paper it is shown that balanced solutions of this problem exist for all values of n which are not powers of a prime.

Key words: combinatorial optimization, turbine balancing problem, balanced solutions, permutations.

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In [2] we proposed a local optimization procedure for solving turbine balancing problem. To compare Mosevich's Monte Carlo method [3] and our local optimization procedure, both were applied to the following artificial *integer turbine balancing problem* $P(n)$:

> Place n points with integer weights $1, 2, 3, \dots, n$ equidistantly on the circle in such a way that their centre of mass is as close to the centre of the circle as possible.

It was surprising that for some values of n it was possible to obtain solutions which are in perfect balance. To get some insight into the nature of the balanced solutions the integer turbine balancing problem was solved by a complete search for n from 3 to 10. There are two balanced solutions (up to rotations and reflections) for $n = 6$ (see Fig. (1) and Fig. (2)):

1 4 5 2 3 6   1 5 3 4 2 6

and 24 balanced solutions for $n = 10$ (the first and the fifth solutions are

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represented in Fig. (3) and Fig. (4)):

1	4	5	8	9	2	3	6	7	10
1	6	3	8	9	2	5	4	7	10
1	7	3	9	5	6	2	8	4	10
1	8	2	9	5	6	3	7	4	10
1	8	4	7	5	6	3	9	2	10
1	9	2	8	5	6	4	7	3	10
2	6	3	9	5	7	1	8	4	10
2	8	1	9	5	7	3	6	4	10
3	2	5	8	9	4	1	6	7	10
3	6	1	8	9	4	5	2	7	10
3	7	1	9	5	8	2	6	4	10
5	2	3	8	9	6	1	4	7	10

1	4	7	6	9	2	3	8	5	10
1	6	7	4	9	2	5	8	3	10
1	7	4	8	5	6	2	9	3	10
1	8	3	6	9	2	7	4	5	10
1	8	5	4	9	2	7	6	3	10
1	9	3	7	5	6	4	8	2	10
2	6	4	8	5	7	1	9	3	10
2	9	1	8	5	7	4	6	3	10
3	2	7	6	9	4	1	8	5	10
3	6	2	9	5	8	1	7	4	10
3	8	1	6	9	4	7	2	5	10
5	4	1	8	9	6	3	2	7	10

The following two balanced solutions for $n = 12$ and $n = 14$

1 5 10 4 11 7 2 6 9 3 12 8
1 14 5 11 2 10 6 8 7 12 4 9 3 13

were obtained by local optimization.

This turns the integer turbine balancing problem into an interesting combinatorial problem. Let us make some observations about it.

There is an interesting reformulation of the integer turbine balancing problem, pointed out by Ante Graovac:

Find all equiangular n-gons with edges of integer lengths from 1 to n.

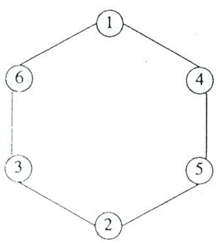

Figure (1)

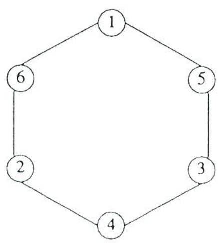

Figure (2)

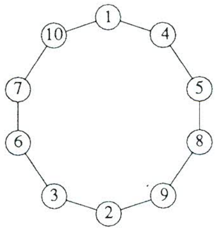

FIGURE (3)

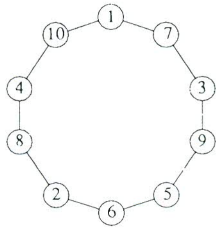

FIGURE (4)

Examples of balanced solutions

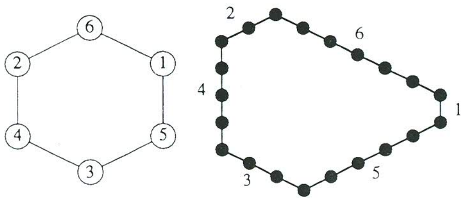

FIGURE (5)

Balanced solution and
corresponding equiangular n-gon

For example, to the balanced solution from Fig. (2) corresponds an equiangular hexagon represented in Fig. (5).

The balanced solutions were obtained by computations with real numbers. Are they truly balanced? To decide this it is sufficient to find two independent axes of balance which intersect in the centre of the circle. For example, in the solution for $n = 10$, represented in Figure 3, the axes 6-5 $(10+1 = 2+9\,,\ 7+4 = 3+8)$ and 2-1 $(7+6 = 5+8\,,\ 10+3 = 4+9)$ are such axes of balance.

For $n = 10$ there are 24 solutions. Are they related? The affirmative answer to this question is given by the following transformation:

PROPOSITION 1 — *Suppose that the points with weights a, b, c and d form a quadrangle in the given balanced solution (see Fig. (6)) and that the equality $a + b = c + d$ holds. Then the solution obtained from it by placing the weights in the vertices of quadrangle in the order d, c, b and a (see Fig. (6)) is also balanced.*

PROOF — Because the initial solution in Fig. (6) is balanced the axes of symmetry s and t are axes of balance. The proposed transformation does not change the corresponding sums on the symmetric lines. Therefore s and t are also axes of balance for the transformed solution in Fig. (6). Thus both solutions have the same centre of mass (the intersection of t and s) which coincides with the centre of the circle: they are both balanced. □

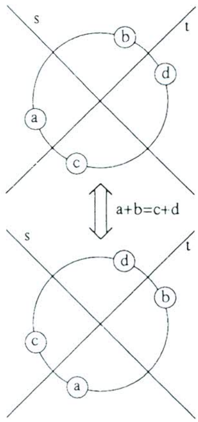

FIGURE (6)
Transformation preserving balanced solutions

Applying this transformation to the solutions for $n = 10$ it turns out that there are only two unrelated (nonequivalent) solutions, represented in Fig. (3) and Fig. (4).

Now we give a formal description of (integer) turbine balancing problem. It is worthwhile starting in a broader framework. Consider a given set $A \subseteq R$ and let

$$M = [m_k : k \in I] \subseteq A\,, \quad I = \{1, \ldots, n\}$$

be a finite multiset of masses. We shall call a *solution* (for M) a multiset

$$\sigma = [(m_k, \varphi_k) : k \in I]$$

where $m_k \in M$ and $\varphi_k \in R$. Additional restrictions on φ_k can be imposed. For example: the solution is *equidistant* if and only if

$$\exists \psi \in R : [(\varphi_k + \psi) \bmod 2\pi : k \in I] = \left[\left(\frac{2\pi j}{n}\right) \bmod 2\pi : j \in I\right].$$

We shall denote by $\mathcal{S}(A)$ the set of all solutions over the set A and by $\mathcal{S}_0(A)$ the corresponding set of equidistant solutions; by $[M]$ we shall restrict the sets to the solutions for M.

To each solution σ we can assign a complex number - the *centre* of σ:

$$c(\sigma) = \sum_{k \in I} m_k e^{i\varphi_k} .$$

The solution σ is *balanced* if $c(\sigma) = 0$. We shall denote by $\mathcal{B}(A)$ the set of all balanced solutions over A and by $\mathcal{B}_0(A)$ the corresponding set of equidistant balanced solutions.

The *turbine balancing problem* can now be expressed as follows:

> For a given multiset of masses M and the corresponding set of solutions $\mathcal{S}[M]$ determine the solution $\sigma^* \in \mathcal{S}[M]$ for which
>
> $$|c(\sigma^*)| = \min\{|c(\sigma)| : \sigma \in \mathcal{S}[M]\} .$$

We shall denote the set of all such solutions by Min $\mathcal{S}[M]$.

We can define some operations on solutions:

$$\begin{aligned} \mathrm{Rot}(\sigma, \psi) &= [(m_k, (\varphi_k + \psi) \bmod 2\pi) : k \in I] \\ a \cdot \sigma &= [(a \cdot m_k, \varphi_k) : k \in I] \\ \mathrm{Inv}(\sigma) &= [(m_k, (2\pi - \varphi_k) \bmod 2\pi) : k \in I] \end{aligned}$$

and two types of reductions:

$$(0, \varphi_k) \in \sigma \Rightarrow \varphi\text{-red } \sigma = \sigma \setminus [(0, \varphi_k)]$$

$$(m_p, \varphi_p), (m_q, \varphi_q) \in \sigma, \ \varphi_p = \varphi_q \bmod 2\pi \Rightarrow$$

$$m\text{-red } \sigma = (\sigma \setminus [(m_p, \varphi_p), (m_q, \varphi_q)]) \cup [(m_p + m_q, \varphi_p)] .$$

We shall denote by m–Red σ the solution obtained from σ by m–reductions which can not be further reduced. This solution is uniquely determined.

Now we can define another operation on solutions:

$$\sigma + \eta = m\text{-Red } (\sigma \cup \eta) .$$

All four operations transform balanced solutions into balanced solutions and, except the sum, equidistant solutions into equidistant solutions. There are several equalities connecting these operations. For example:

$$\begin{aligned} a \cdot (\sigma_1 + \sigma_2) &= a \cdot \sigma_1 + a \cdot \sigma_2 \\ \mathrm{Rot}(\sigma_1 + \sigma_2, \psi) &= \mathrm{Rot}(\sigma_1, \psi) + \mathrm{Rot}(\sigma_2, \psi) \\ \mathrm{Rot}(\mathrm{Inv}(\sigma), \psi) &= \mathrm{Inv}(\mathrm{Rot}(\sigma, -\psi)) . \end{aligned}$$

Several types of equivalences and canonical solutions can also be introduced.

It is easy to prove the following:

PROPOSITION 2 — *The solution*

$$\epsilon(n) = \left[\left(1, \frac{2\pi k}{n}\right) : k \in I\right], \quad n > 1$$

is balanced and equidistant.

PROPOSITION 3 — *Let σ be any solution of the turbine balancing problem then*

$$\sigma' = \sum_{s=0}^{q-1} Rot\ \left(\sigma, \frac{2\pi s}{q}\right)$$

is a balanced solution.

There is a simple construction for a balanced solution of $P(n)$ for n of the form $4k + 2$ (see Fig. (7), (8), (9)).

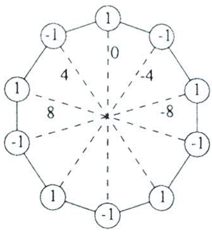

FIGURE (7)

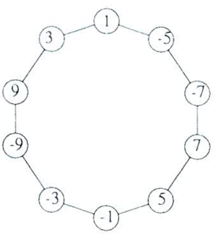

FIGURE (8)

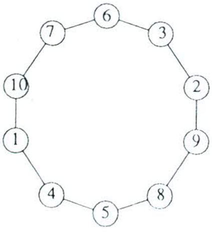

FIGURE (9)

Construction for $n = 4k + 2$

$$\sigma_a = \epsilon(2k+1) + \text{Rot}\left((-1)\cdot\epsilon(2k+1), \frac{2\pi}{n}\right)$$

$$\sigma_b = \sigma_a + \sum_{s=-k}^{k} \text{Rot}\left((4s)\cdot\epsilon(2), \frac{2\pi s}{n}\right)$$

$$\sigma_c = \frac{1}{2}\cdot(\sigma_b + (n+1)\cdot\epsilon(n)).$$

These solutions can also be represented, as pointed out by Branislava Peruničić, as a Möbius ladder graph (see Fig. (10)).

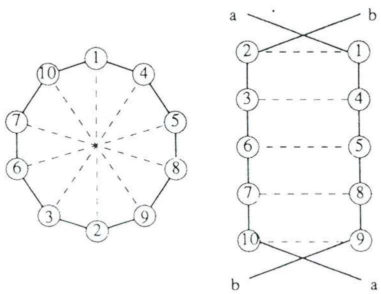

FIGURE (10)
Mobius ladder

PROPOSITION 4 — *Let n be an even number, $n \neq 2^k$. Then the problem $P(n)$ has a balanced solution.*

PROOF — We can write the number n in the form $n = 2^p(2m+1)$. There are two cases to consider:

Case A. $p = 1$; the balanced solution can by obtained by a construction previously described.

Case B. $p > 1$; in this case we can write n in the form $n = 2^{p-1}2(2m+1)$. Let us denote $u = 2^{p-1}$ and $v = 2(2m+1)$. Then by a previous construction there exists a balanced solution σ of the problem $P(v)$ and therefore

$$\sigma' = \sum_{s=0}^{u-1} \text{Rot}\left(\sigma + (sv)\cdot\epsilon(v), \frac{2\pi s}{n}\right)$$

is also a balanced solution of the problem $P(n)$. □

On the basis of computer results for small n we conjectured that there is no balanced solution of the problem $P(n)$ for n odd. This conjecture was shown false by Janez Žerovnik who found a balanced solution for $n = 15$ (see Fig. (11), (12)):

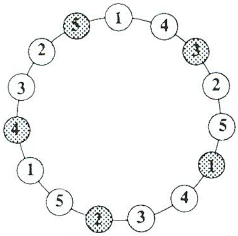

FIGURE (11)

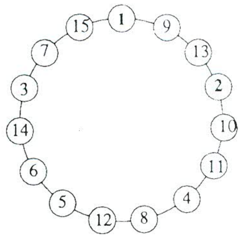

FIGURE (12)

Construction of balanced solution for $n = 15$

$$\sigma = 1\,2\,4\,5\,3 \in S_0$$

$$\sigma_a = \sigma + \text{Rot}\left(\sigma, \frac{2\pi}{3}\right) + \text{Rot}\left(\sigma, -\frac{2\pi}{3}\right)$$

$$\sigma_b = \sigma_a + \text{Rot}\left(5 \cdot \epsilon(5), \frac{2\pi}{3}\right) + \text{Rot}\left(10 \cdot \epsilon(5), -\frac{2\pi}{3}\right)$$

This solution can be generalized to the following theorem.

THEOREM 1 — *Let n have the form $n = pq$, $p, q > 1$, $\gcd(p, q) = 1$. Then the problem $P(n)$ has a balanced solution.*

PROOF — Let $\sigma \in S$ be any solution of the problem $P(p)$. Then the solution

$$\sigma' = \sum_{s=0}^{q-1} \text{Rot}\left(\sigma + (sp) \cdot \epsilon(p), \frac{2\pi s}{q}\right)$$

is a balanced solution of the problem $P(n)$.

To see this we rewrite σ' in the form

$$\sigma' = \sum_{s=0}^{q-1} \text{Rot}\left(\sigma, \frac{2\pi s}{q}\right) + \sum_{s=0}^{q-1} \text{Rot}\left((sp) \cdot \epsilon(p), \frac{2\pi s}{q}\right).$$

The first term is balanced by Proposition 3; the second term is composition of balanced components. □

This theorem tells us that the problem $P(n)$ has a balanced solution for all n which are not powers of some prime number. Marko Petkovšek and Aleksander Malnič noticed that if we associate to the centre of an equidistant solution σ

$$c(\sigma) = \sum_{k \in I} m_k e^{i\frac{2\pi k}{n}}, \quad I = \{0, \dots, n-1\}$$

a polynomial

$$p(x; \sigma) = \sum_{k \in I} m_k x^k$$

then for balanced solutions this polynomial is divisible by the minimal polynomial corresponding to the n-th root of unity. This property can be used to prove the final result:

THEOREM 2 — *There is no balanced solution of the problem $P(n)$ for $n = p^m$, p prime.*

The detailed proof and some other results will be published in a separate paper [1].

Let us conclude with some questions.

1) Are there other ways to transform or combine (balanced) solutions into balanced solutions?
2) What is the complexity of the integer turbine problem; is there an efficient procedure to count/generate all balanced solutions of $P(n)$?
3) Given a multiset of masses M, decide whether $P(M)$ has a balanced solution!

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