

## INDUCTIVE CLASSES OF CUBIC GRAPHS

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### ABSTRACT

The inductive definitions of the following classes of (simple) cubic graphs are given:

- (all) cubic graphs,
- connected cubic graphs,
- 2-connected cubic graphs,
- planar cubic graphs,
- connected planar cubic graphs and
- 3-connected planar cubic graphs.

There are two ways in mathematics to define an infinite class:

- by listing the properties which every object belonging to the class has to satisfy;
- by describing how the objects belonging to the class can be built from a given class of basic objects.

In the discrete/algorithmic approach we speak in the first case about the *recognition* of objects from the class and in the second case about the

generation of objects from the class. The class of objects which can be obtained with generation is often called *inductive class*.

Following Curry [6] an inductive class  $I$  is defined by giving:

- *initial specifications*; which define the class  $B$  of initial objects – the *basis* of  $I$ ;
- *generating specifications*; which define the class  $R$  of *rules* (modes) of combination – any such rule applied to an appropriate sequence of objects, already in  $I$ , produces an object of  $I$ .

The inductive class  $I = \text{Cn}(B, R)$  consists exactly of the objects which can be obtained (constructed) from the basis by a finite number of applications of the generating rules.

A powerful proof technique for the properties of objects of the inductive class is the *inductive generalization* (structural induction): in order to show that every object from  $I$  has a certain property  $P$ , it is sufficient to establish that:

- every object of the basis has the property  $P$ ;
- the generating rules preserve the property  $P$ .

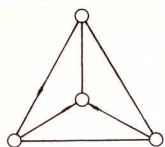
Another useful property of inductive classes is expressed by the following proposition:

**Lemma 1.** *Let  $I = \text{Cn}(B, R)$  and  $I' = \text{Cn}(B', R')$  be inductive classes, such that  $B' \subseteq I$  and the generating rules  $R'$  can be deduced in  $I$ . Then  $I' \subseteq I$ .*

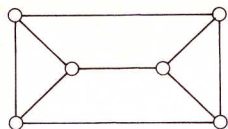
In this paper the inductive definitions of some classes of (simple) cubic graphs are given. We shall denote with  $C_i$  ( $i = 0, 1, 2, 3$ ) the class of  $i$ -connected cubic graphs and with  $C_{Pi}$  ( $i = 0, 1, 2, 3$ ) the class of  $i$ -connected planar cubic graphs.

**Theorem.** *If we label the following graphs;*

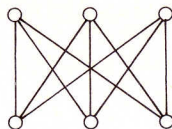
B1.



B2.

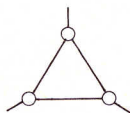


B3.

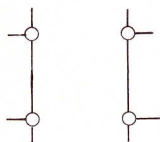
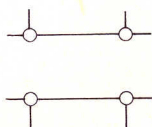


and the following generating rules:

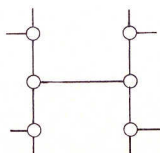
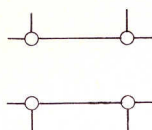
P1.



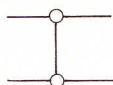
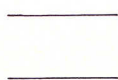
P2.



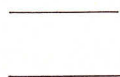
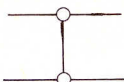
P3.



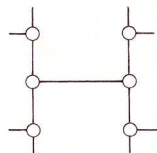
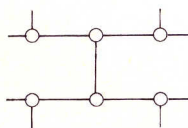
P4.



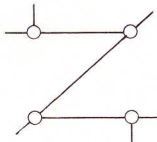
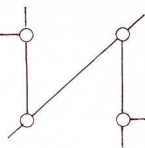
P5.



P6.



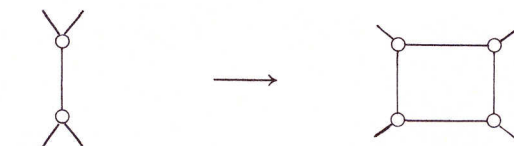
P7.



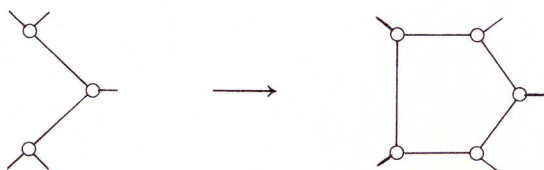
P8.



P9.



P10.



then

$$T1. \quad Cn(B1; P1, P6, P5) = C_0$$

$$T2. \quad Cn(B1; P1, P7, P5) = C_0$$

$$T3. \quad Cn(B1; P1, P2) = C_0$$

$$T4. \quad Cn(B1; P4, P5) = C_0$$

$$T5. \quad Cn(B1; P1, P6) = C_1$$

$$T6. \quad Cn(B1; P1, P7) = C_1$$

$$T7. \quad Cn(B1, B2, B3; P3) = C_1$$

$$T8. \quad Cn(B1; P4, P8) = C_2$$

$$T9. \quad Cn(BL1; PL1, PL6, PL5) = C_{P0}$$

$$T10. \quad Cn(BL1; PL1, PL7, PL5) = C_{P0}$$

$$T11. \quad Cn(BL1; PL1, PL6) = C_{P1}$$

$$T12. \quad Cn(BL1; PL1, PL7) = C_{P1}$$

$$T13. \quad Cn(BL1; PL1, PL9, PL10) = C_{P3}$$

where  $L$  in the basic graph or rule label means that the corresponding graph/rule is embedded in the plane; the strict combinatorial description of these rules can be achieved by attaching a rotation to each vertex.

In the generating rules some of the edges (having only one indicated endpoint) may coincide. The applications of the rules have to preserve the simplicity of graphs.

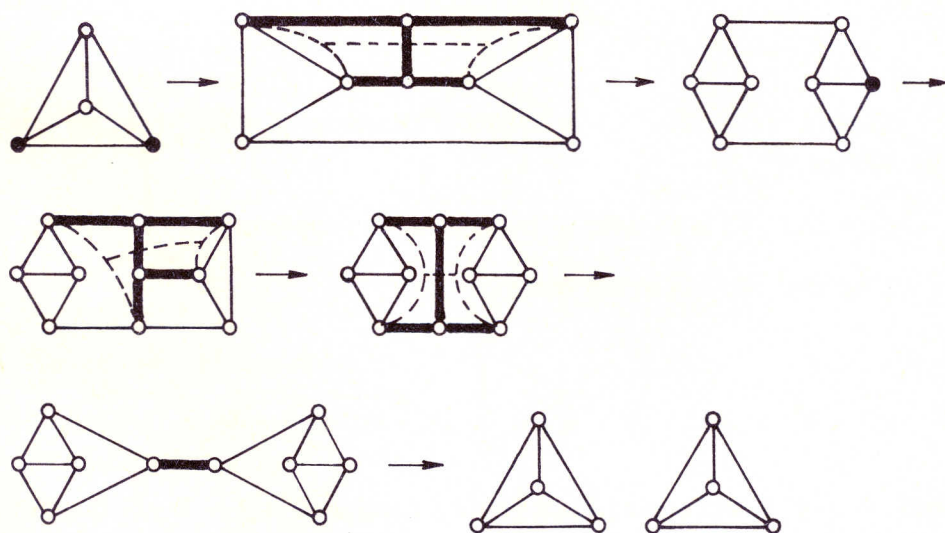
## PROOF OF THEOREM

T1. Because the basic graph  $B1 \cong K_4$  is cubic and the generating rules P1, P6 and P5 preserve cubicity

$$I_1 = \text{Cn}(B1; P1, P6, P5) \subseteq C_0$$

holds, by inductive generalization.

To prove that also  $C_0 \subseteq I_1$ , let us start by showing that the basis of the inductive class  $I_1$  is self-reproducible:



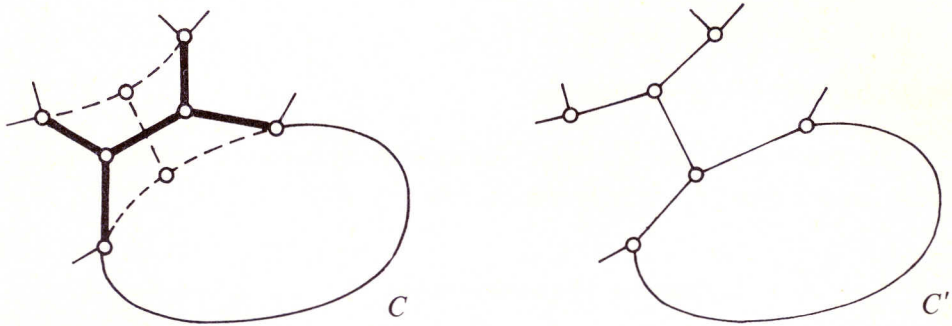


The self-reproducibility permits us to consider in the continuation of the proof only connected (components of) cubic graphs.

Therefore, to show  $C_0 \subseteq I_1$  it is enough to show that every connected cubic graph belongs to  $I_1$ , or equivalently, that every connected cubic graph, different from  $K_4$ , can be reduced (number of vertices) using the inverse rules of P1 and P6.

Let  $G$  be any connected cubic graph. There are two possible cases, depending on the length of the shortest cycle  $C$  in  $G$ :

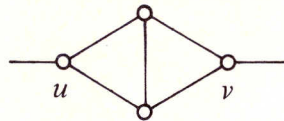
*Case I.*  $|C| > 3$ : we reduce it to case II by repeatedly applying P6<sup>-</sup>:



*Case II.*  $|C| = 3$ :  $C$  is a "triangle". Because  $G \neq K_4$  there are two possibilities:

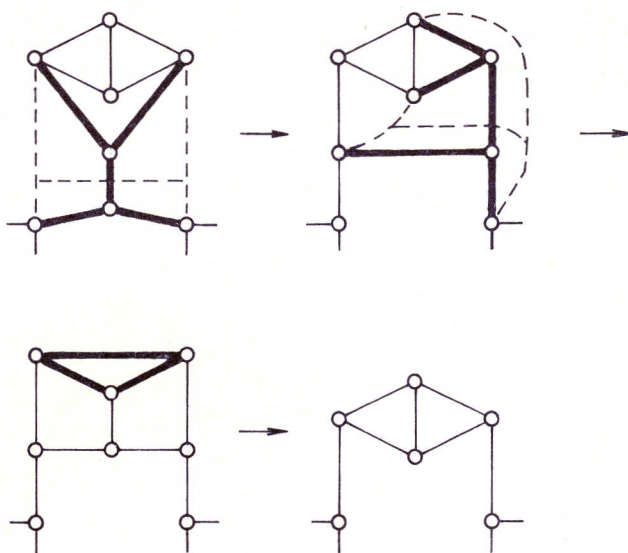
*Case II/1.*  $C$  is an isolated triangle: we can apply rule P1<sup>-</sup>.

*Case II/2.*  $C$  has a twin triangle:

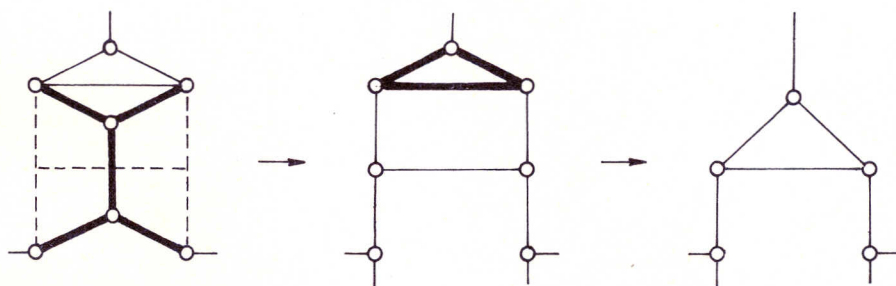


Here  $u$  and  $v$  are non-adjacent vertices because  $G \neq K_4$ . We distinguish between the following two cases whether  $u$  and  $v$  have a third common neighbour or not.

Case II/2.1.



Case II/2.2.



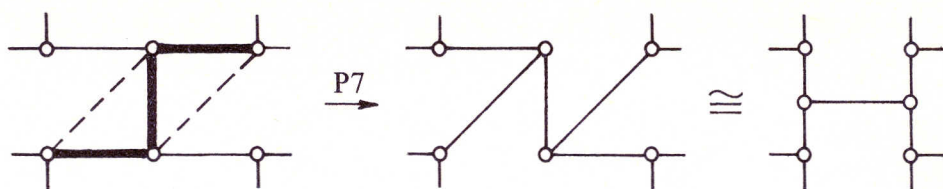
This completes the proof.

Note. Another way to show  $I_1 = C_0$  is based on the fact that rule P3 is deducible in  $I_5$ .

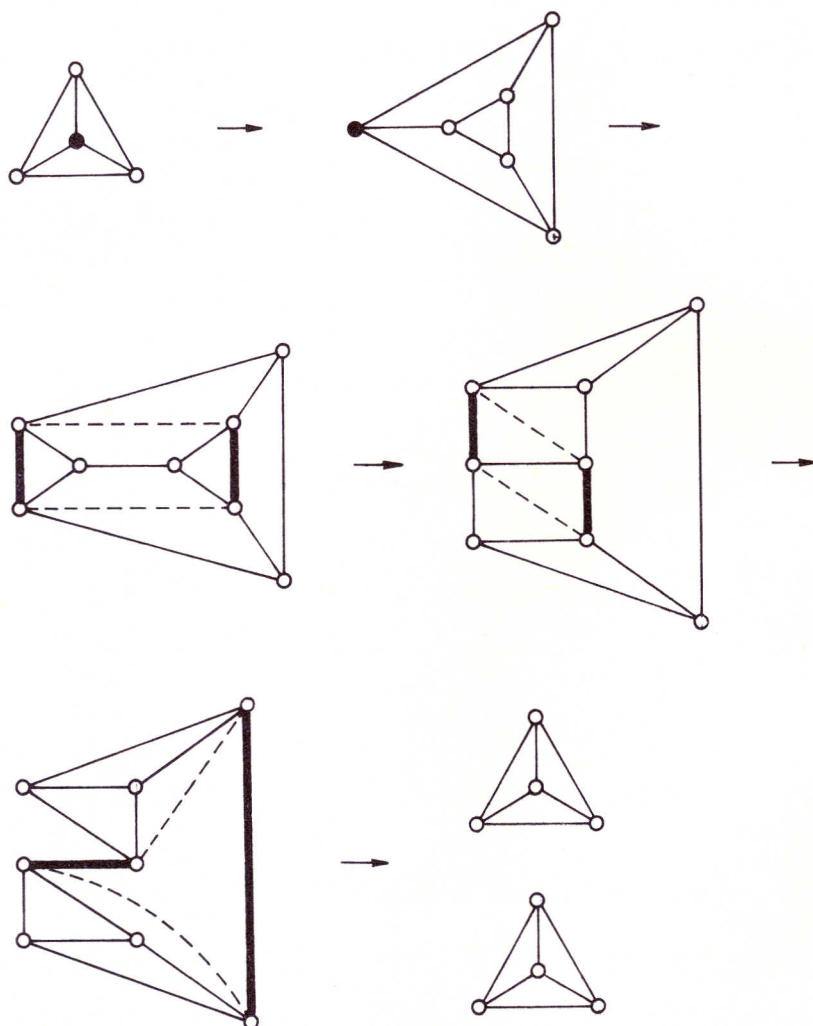
T2. By inductive generalization and T1:

$$I_2 = \text{Cn}(\text{B1}; \text{P1}, \text{P7}, \text{P5}) \subseteq C_0 = I_1.$$

The opposite inclusion  $I_1 \subseteq I_2$  holds by Lemma 1, because rule P6 is deducible in  $I_2$ :



T3. By inductive generalization  $I_3 \subseteq C_0$ . The basis of  $I_3$  is self-reproducible:



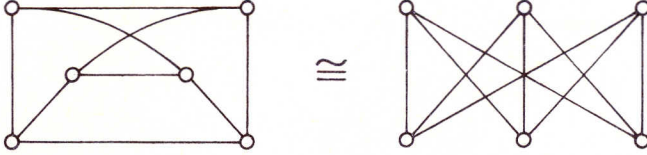


Therefore, to show  $C_0 \subseteq I_3$ , it is enough to show  $C_1 \subseteq I_3$ . This follows by Lemma 1 and T7 because  $B(I_7) \subseteq I_3$ :

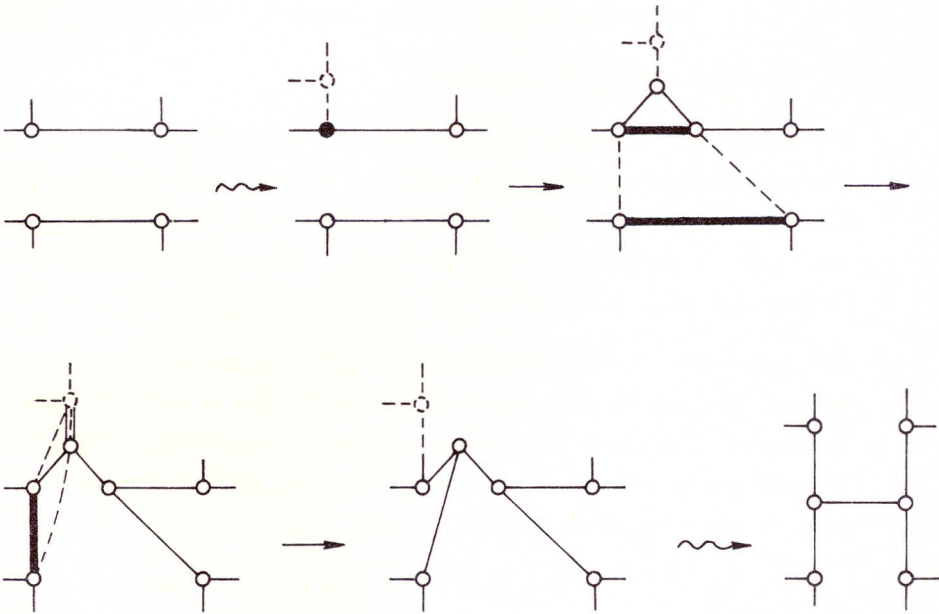
B2:



B3:

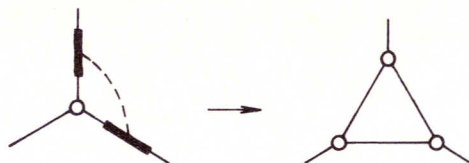


and P3 is deducible in  $I_3$ :

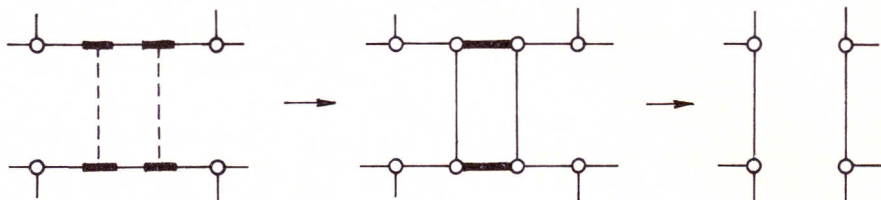


T4. By inductive generalization  $I_4 \subseteq C_0$ ; and  $C_0 = I_3 \subseteq I_4$  by Lemma 1, because the rules P1 and P2 are deducible in  $I_4$ :

P1:



P2:



T5. See T1.

T6. See T2 and T1.

T7. (Johnson's definition) See [13], p. 129–133.

T8.  $I_8 \subseteq C_2$  follows by inductive generalization, because:

- $K_4$  (basic graph) is 2-connected;
- P4 preserves 2-connectedness: the old vertices remain 2-connected, because no edge was removed; the new pair of vertices is 2-connected, because each pair of edges lies on a common cycle; each pair of an old and a new vertex is 2-connected, because each vertex and each edge belong to a common cycle (see [8], p. 65);
- P8 preserves 2-connectedness: the old vertices remain 2-connected, because no edge was removed; each pair of new vertices is 2-connected, because they lie on a common cycle; each pair of an old and a new vertex is 2-connected, because each vertex and each edge belong to a common cycle.

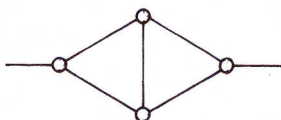
To show that  $C_2 \subseteq I_8$  we shall show that each 2-connected cubic graph  $G$ , different from  $K_4$ , can be reduced (number of vertices) using the inverse rules of P4 and P8. To this purpose we have to examine two cases:

*Case I.* The graph  $G \neq K_4$  contains a triangle. There are two possibilities:

*Case I/1.* If the triangle is isolated then we can apply  $P4^-$

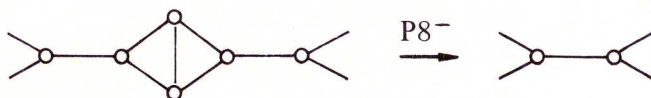


*Case I/2.* The triangle has a twin:

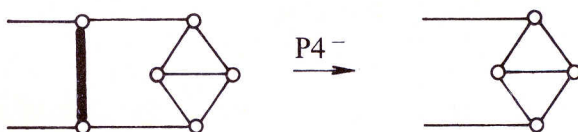


This configuration has (because  $G$  is 2-connected) two possible extensions, which can be reduced as follows:

*Case I/2.1.*



*Case I/2.2.*



*Case II.* There is no triangle in graph  $G$ . It follows from exercise 6.33 of [10], that in  $G$  there exists an edge to which the rule  $P4^-$  can be applied without destroying the 2-connectedness and simplicity of the graph.

T9. See T1. The proof of T1 is essentially planar.

T10. See T2 and T1.

T11. See T1.

T12. See T2 and T1.

T13. See [7].

Here follow some open questions for future research in inductive definitions (of cubic graphs):

- find the inductive definitions of  $C_3$  and  $C_{P_2}$ !
- the generating rule is said to be *local* if its left side consists of a connected part of the graph. The rules P2, P3 and P4 are not local. Find the local inductive definition of  $C_2$  or prove that no such definition exists!
- can the inductive definitions be "married" with orderly algorithms [5]?

For inductive definitions of other classes of graphs see [1], [2], [3], [4], [11], [12], [14].

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