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Symmetric-Acyclic Decompositions of Networks

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Abstract: This paper presents two new developments for partitioning networks. One is the symmetric-acyclic decompostition of a network into clusters of vertices where the vertices in a cluster are linked only by symmetric ties only (with null ties for some pairs of vertices permitted). The induced structure of clusters and ties between clusters is an acyclic graph. A corresponding ideal blockmodel is defined and, given this definition, a generalized blockmodeling method for establishing such decompositions of networks is the second approach introduced here. Both are founded in the Davis and Leinhardt (1972) formulation of a ranked clusters model as a theoretical expectation concerning the structure of human groups and directed affect ties. The decomposition also creates a delineation of the internal structure of identified components but is sensitive to departures from the ideal model. The generalized blockmodeling approach is complementary to the decomposition because it is robust in the presence of such departures and, moreover, identifies them. While initially formulated in a small groups context, the ranked clusters model can be applied to a variety of network phenomena. We illustrate the decomposition and generalized blockmodeling methods with the marriage network of noble families in Ragusa (Dubrovnik) for the 18th Century and early 19th Century.

Keywords: Social networks; Partitioning; Blockmodels.

1. Introduction

Our concern here is the partitioning of social networks into clusters (called positions) and the delineation of the structure of the whole network in terms of the identified positions[1] and the relations between them. The reference point for these partitions is an 'ideal' structure where the ties between members of a position are symmetric (with null ties permitted[2]) and the network of the positions and the ties between positions is an acylic graph.

The ideal model is grounded in substance and stems from the work of Davis and Leinhardt (1972) who distilled some essential ideas from Homans (1950, pp 108-130; 140-147) and formulated a ranked clusters model, one that results from the operation of two sets of social forces. This resulting model — for small groups of social actors and an affect tie — has two distinct structural features. One is the differentiation of the small group into cliques (in the sense of a maximal complete subgraph) while the other is the elaboration of 'ranks' that can be described in terms of levels. The social processes generating these two structural features reinforce each other with the result that the cliques are distributed across the levels.

1. We use the term 'location' to describe the pattern of ties for an actor (vertex) and the term 'position' for a cluster of vertices that has been identified in some fashion.

2. In fact, they are a special case of a symmetric tie.

2. Acyclic Structures

The initial formulation of the ranked clusters model was for a directed affect tie such as 'likes'. We describe the tie between pairs of actors who like each other as *symmetric* or mutual. When one actor likes another actor and the sentiment is not reciprocated, the tie is *asymmetric*. One part of the ideal structure is the formation of cliques with the mutual ties and the second is the ranking generated by the asymmetric ties. The presumption is that the asymmetric ties are directed in a systematic fashion. In the small group context, the more popular actors are ranked higher and the asymmetric ties are directed 'upwards'. When they exist, the asymmetric liking ties go from lower ranked actors to higher ranked actors. Later, we define the term 'level' precisely. For our purposes now, the intuitive idea of there being levels is enough. The case where a clique exists as the sole occupant of a level is straightforward. When multiple cliques do so, there are no ties between members of different cliques on the same level. We emphasise that this is a specification of an *ideal* structure.

This partition of the ties into symmetric and asymmetric parts is central to the ranked clusters model. For cliques at distinct levels, the hypothesized structure has asymmetric ties directed 'up' from lower ranked cliques to higher ranked cliques. In this ideal structure, no asymmetric ties are directed 'down' from members of higher ranked cliques to actors in lower level cliques. Nor are there mutual ties between units at different levels. In general, the direction of the relation is arbitrary. If instead of 'likes', we used 'is liked by', the ties would be directed down and the forbidden direction for the ties would be up. Later, we argue that the ranked clusters model is not restricted to small groups. If the relation is 'has authority over' the permitted direction is down and the forbidden direction for a ranked clusters model is up. It is the reverse for 'is responsible to'. All that matters is that there is a direction (up or down). The specific direction as arbitrary. The acyclic requirement is that the asymmetric ties go in one direction across ranks.

The 'ideal' model used by Davis and Leinhardt (1972) as an illustration of their ranked clusters model is shown in Figure 1 with five positions (as subgroups) and three levels. The highest ranking position is A. Some actors from all other positions direct ties towards actors in A. There are two positions, B and C, at the second level. Some actors in these positions send ties to actors in A and receive ties from actors in D and E. There are no ties between actors in B and C. There are two positions, D and E, in the third (bottom) level whose members direct ties up to actors in all of the other positions. There are no ties between D and E.

In our generalization of the ranked clusters model, we do *not* require subgroups to be maximal complete subgraphs. There are two reasons for this.

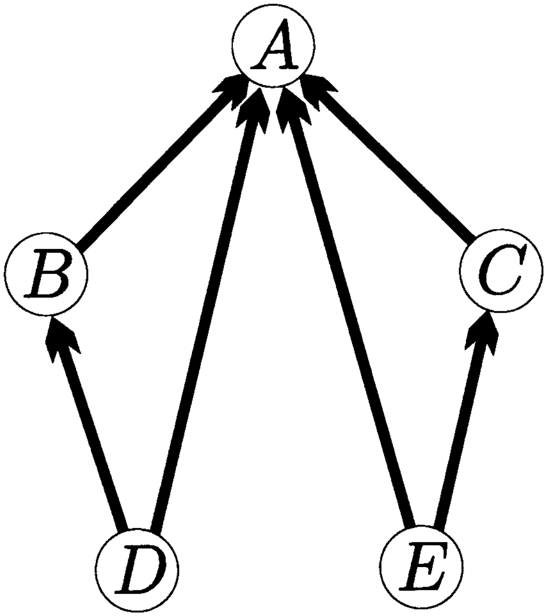

Figure 1. A Three Level Acyclic Model.

Table 1. A Relation R

	a	b	c	d	e	j	k	l	m	n	o	f	g	h	i	p	q	r	s
a	·	1	1	·	·	·	·	·	·	·	·	·	[**1**]	·	·	·	·	·	·
b	1	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
c	1	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
d	·	1	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·
e	·	·	1	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
j	·	·	·	1	·	·	(*1*)	1	·	·	·	·	·	·	·	·	·	·	·
k	·	·	·	·	1	·	·	1	·	·	·	·	·	·	·	·	·	·	·
l	·	·	·	·	1	1	1	·	·	·	·	·	·	·	·	·	·	·	·
m	·	·	1	1	·	·	·	·	·	1	1	·	·	·	·	·	·	·	·
n	·	·	1	·	1	·	·	·	1	·	·	·	·	·	·	·	·	·	·
o	·	·	·	·	·	·	·	·	1	(*1*)	·	·	·	·	·	·	·	·	·
f	·	·	1	·	·	·	·	·	·	·	·	·	1	·	·	·	·	·	·
g	1	·	·	·	·	·	·	·	·	·	·	1	·	1	1	·	·	·	·
h	·	1	·	·	·	·	·	·	·	·	·	·	1	·	1	·	·	·	·
i	·	·	·	·	·	·	·	·	·	·	·	·	1	1	·	·	[**1**]	·	·
p	·	·	·	·	1	·	·	·	·	1	·	·	·	1	·	·	1	·	·
q	·	·	·	1	·	·	·	·	·	·	·	·	·	·	1	1	·	1	1
r	·	·	·	·	·	·	·	·	·	·	1	·	·	·	·	(*1*)	1	·	1
s	·	·	·	·	·	·	·	·	·	·	·	·	1	·	·	·	1	1	·

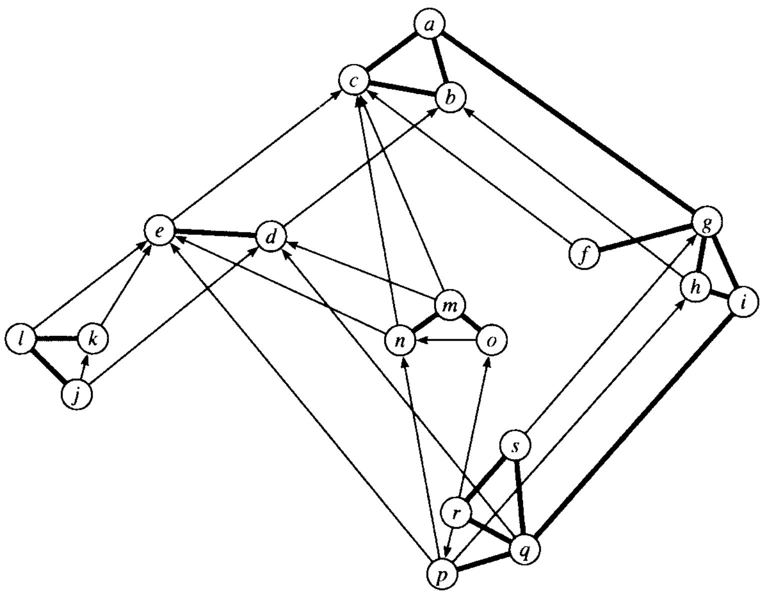

Figure 2. Graph of the Relation in Table 1.

One is the widely accepted argument that cliques are too stringent because they require the presence of all ties for a cohesive subgroup — hence, for example, the efforts to specify *n*-cliques (Luce 1950), *n*-clans (Mokken 1979) and *k*-plexes (Seidman and Foster 1978). A second reason for liberalizing the clique idea is found in the idea that these structures are generated by processes and that the observed structures are not likely to be in an equilibrium state. The process generating cohesive subgroups may be incomplete and not all mutual ties have been generated by the time the structure is observed. We note that, in this generalization of the ranked cluster structure, a pair of cliques on the same level can be connected by a symmetric tie.

Table 1 and Figure 2 contain a constructed hypothetical example that we use to illustrate the symmetric-acyclic decomposition described in the next section and the generalized blockmodeling partitioning method. In Table 1, the period '·' represents a 0. The italicized 1's which are circled (i.e., $j \rightarrow k$, $o \rightarrow n$, and $r \rightarrow p$) are inconsistent with symmetry inside clusters. Bold faced 1's which are shown in squares (i.e. $a \rightarrow g$, and $i \rightarrow q$) represent ties that are inconsistent with the assumed acyclic structure. In Figure 2, the thick ties represent symmetric ties and the thin lines represent asymmetric ties for the symmetric-acyclic decomposition and the generalized blockmodeling method.

3. Establishing Symmetric-Acyclic Decompositions of Networks

We present a new tool, called the symmetric-acyclic decomposition method, for verifying the presence (or not) of an acyclic structure characteristic of the ranked clusters model. The networks considered here have directed ties. The 'pure' ranked clusters model is an acyclic directed graph. Any departures from such a structure are not allowed.

Let $\mathbf{U} = \{x_1, x_2, \ldots, x_n\}$ be a finite set of actors or *units*. The units are related by a binary *relation*

$$R \subseteq \mathbf{U} \times \mathbf{U}$$

which determines a *network*

$$\mathbf{Net} = (\mathbf{U}, R)\,.$$

The network shown in Figure 2 is a graph with units as vertices.

A *clustering* $\mathbf{C} = \{C_1, C_2, \ldots, C_k\}$ is a partition of $\mathbf{U}$ where $C_i \subseteq \mathbf{U}$ are *clusters*. $\mathbf{C}$ partitions the relation R into *blocks*

$$R(C_i, C_j) = R \cap C_i \times C_j\,.$$

Each such block consists of units belonging to clusters C_i and C_j and all the arcs leading from cluster C_i to cluster C_j. If $i = j$, a block $R(C_i, C_i)$ is called a *diagonal* block.

We say that a clustering $\mathbf{C}$ over the relation R is a *symmetric-acyclic clustering* iff (a) all subgraphs induced by clusters from $\mathbf{C}$ contain only bidirectional arcs (*edges*); and (b) each closed walk in $(\mathbf{U}, R)$ is entirely contained in a single cluster of $\mathbf{C}$.

3.1 Ideal Structures

The vertices are partitioned into positions based on the pattern of ties linking them. For the ideal ranked clusters structure, there is also a partition of the ties into symmetric and asymmetric ties. The symmetric ties are found only inside the diagonal blocks and the asymmetric ties are located in blocks above or below the diagonal (but not both) depending on the intrinsic direction of the relation studied. We express this as decomposing a relation R into two relations, S and Q, as follows.[3]

3. We use ^{T} to denote the transpose, consistent with a widely used convention. Later we use the symbol $'$ to label a factor set $\mathbf{U}'$ (which is the set of clusters generated in the decomposition).

A relation $R \subseteq \mathbf{U} \times \mathbf{U}$ has a *symmetric-acyclic decomposition* (S,Q) iff there exist relations $S,Q \subseteq \mathbf{U} \times \mathbf{U}$ such that (1) (S,Q) is a partition of R, $S \cup Q = R$ and $S \cap Q = \emptyset$, (2) S is symmetric, $S = S^T$, and (3) Q is acyclic in $(\mathbf{U},R)$, $S^* \cap Q * R^* = \emptyset$, where R^* is the transitive and reflexive closure of relation R, and the operation $*$ denotes the product of (any two) relations R_1 and R_2: $R_1 * R_2 = \{(x,y) : \exists z : (xR_1z \wedge zR_2y)\}$. For Theorem 1, we use the notation $R \setminus S$ to denote the elements of the set R that do not belong to S.

Theorem 1. *If a relation $R \subseteq \mathbf{U} \times \mathbf{U}$ has a symmetric-acyclic decomposition (S,Q) then*

$$S = R \cap R^T \quad \text{and} \quad Q = R \setminus S\,.$$

The reflexive and transitive closure S^* of relation S is an equivalence relation, equal to the strong connectivity relation in $(\mathbf{U},R)$. Let $\mathbf{U}'$ denote the factor set $\mathbf{U}/S^*$. $\mathbf{U}'$ is a symmetric-acyclic clustering. The relation $\sqsubset$ defined on $\mathbf{U}'$ by

$$X \sqsubset Y \equiv \exists x \in X,\ \exists y \in Y : xQy$$

is acyclic and irreflexive.

If the relation shown in Table 1 had 0's instead of the italicized 1's and the bold faced 1's, it would have a symmetric-acyclic decomposition. As such, it is an ideal relation and we confine attention to it in this section. The ideal ranked clusters structure for this relation is drawn on the top of Figure 3 with six positions: $P_1 = \{a,b,c\}$, $P_2 = \{d,e\}$, $P_3 = \{j,k,l\}$, $P_4 = \{m,n,o\}$, $P_5 = \{f,g,h,i\}$, and $P_6 = \{p,q,r,s\}$.

We can assign levels (or ranks) to positions from $\mathbf{U}'$ by defining a *level* function: $h : \mathbf{U}' \to 1, \ldots, L_h$ satisfying the conditions (a) h is surjective, and (b) for each pair of positions, $P_i, P_j \in \mathbf{U}'$, $P_i \sqsubset P_j \Rightarrow h(P_i) < h(P_j)$.

Each level function determines a symmetric-acyclic clustering $\mathbf{C} = \{C_i\}$ where

$$C_i = \bigcup_{P \in \mathbf{U}' : h(P) = i} P$$

and vice versa

$$P_i, P_j \in C \in \mathbf{C} \Rightarrow h(P_i) = h(P_j)\,.$$

In other words, clusters in a symmetric-acyclic clustering are exactly unions of the positions at the same level.

The *cover* relation $\sqsubset\!\!\cdot$, analogous to the Hasse relation of an order, can be defined also for an acyclic relation $\sqsubset$ by

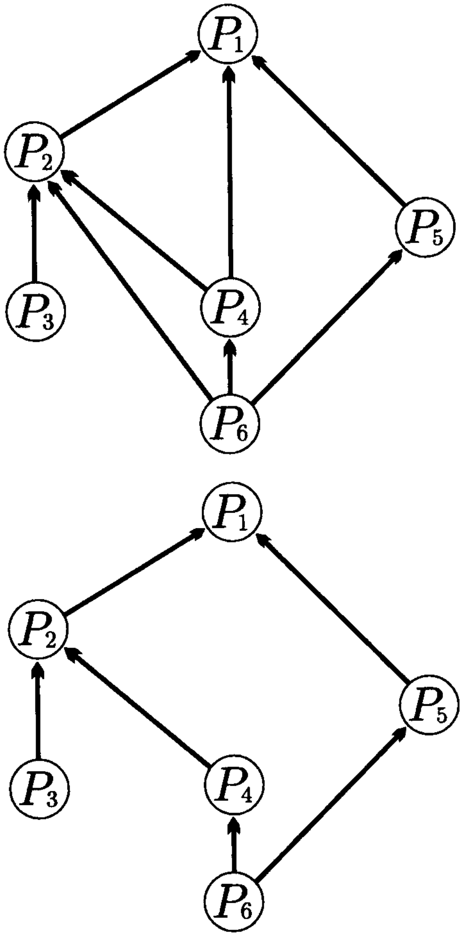

Figure 3. A Four Level Acyclic Graph for Relation in Table 1 and its Hasse Graph.

$$\dot{\sqsubset} = \sqsubset \setminus \sqsubset * \overline{\sqsubset}$$

where $\overline{\sqsubset}$ denotes the transitive closure of relation $\sqsubset$.

We obtain $\dot{\sqsubset}$ from $\sqsubset$ by deleting each arc (x,y) with the property that there exists a path of length at least 2 from x to y. Using the Hasse graph $\mathbf{H} = (\mathbf{U}', \dot{\sqsubset})$ we can describe the set Σ of all acyclic clusterings with symmetric clusters over R: (a) the minimum number of clusters is equal to $1 + d$, where d is the length of the longest path in $\mathbf{H}$ (and implies min $L_h = 1 + d$); (b) the maximum number of clusters is equal to card $\mathbf{U}'$ (and implies max L_h = card ($\mathbf{U}'$); (c) card $\Sigma = 1$ iff $\mathbf{H}$ is a path; and (d) each k-clustering (partition into k clusters) is determined by a k-set (a set with k elements) of "compatible independent" vertices/positions of $\mathbf{H}$.

Two vertices $u,v \in \mathbf{U}'$ are *independent* iff they are not connected by a path in $\mathbf{H}$. A pair of sets $X,Y \subset \mathbf{U}'$ is *compatible* iff

$$\neg\, \exists x_1,x_2 \in X,\ \exists y_1,y_2 \in Y : (x_1 \sqsubset y_1 \wedge y_2 \sqsubset x_2)\,.$$

All pairs of sets in a k-set have to be compatible. The clusters are unions of the sets corresponding to vertices from the sets of a k-set. We note that, as a special case, an asymmetric tie down a ranked clusters model leads to an incompatibility.

For the idealized relation shown in Table 1, the graph of the relation $\sqsubset$ is presented on the top panel of Figure 3. To obtain the Hasse graph $\mathbf{H}$ (see the bottom of Figure 3) two arcs (P_4,P_1) (given the path $P_4P_2P_1$) and (P_6,P_2) (given the path $P_6P_4P_2$) have to be deleted. The longest path in $\mathbf{H}$ has length 3 and therefore the number of levels is at least 4. There are further symmetric-acyclic clusterings that can be obtained from P_1 through P_6 providing compatibility is preserved. There are 10 such clusterings and we denote them with $\mathbf{C}^i_j$ where i is the number of clusters and j is an index running over the clusterings with a given number of clusters. Thus $\mathbf{C}^4_1$ is the first such clustering with four clusters. There are four symmetric-asymmetric clusters with four clusters, five with 5 clusters and one partition with 6 clusters. From $\mathbf{H}$ we see that the following symmetric-acyclic clusterings (where only compatible clusters can be grouped together) exist:

$$\begin{aligned}
\mathbf{C}^4_1 &= \{P_1, P_2 \cup P_5, P_3 \cup P_4, P_6\}\\
\mathbf{C}^4_2 &= \{P_1, P_2, P_3 \cup P_4 \cup P_5, P_6\}\\
\mathbf{C}^4_3 &= \{P_1, P_2 \cup P_5, P_4, P_3 \cup P_6\}\\
\mathbf{C}^4_4 &= \{P_1, P_2, P_4 \cup P_5, P_3 \cup P_6\}\\
\mathbf{C}^5_1 &= \{P_1, P_2 \cup P_5, P_3, P_4, P_6\}\\
\mathbf{C}^5_2 &= \{P_1, P_2, P_3, P_4 \cup P_5, P_6\}\\
\mathbf{C}^5_3 &= \{P_1, P_2, P_3 \cup P_5, P_4, P_6\}\\
\mathbf{C}^5_4 &= \{P_1, P_2, P_3 \cup P_6, P_4, P_5\}\\
\mathbf{C}^5_5 &= \{P_1, P_2, P_3 \cup P_4, P_5, P_6\}\\
\mathbf{C}^6_1 &= \{P_1, P_2, P_3, P_4, P_5, P_6\} = \mathbf{U}'\,.
\end{aligned}$$

We can introduce, in Σ, an operation

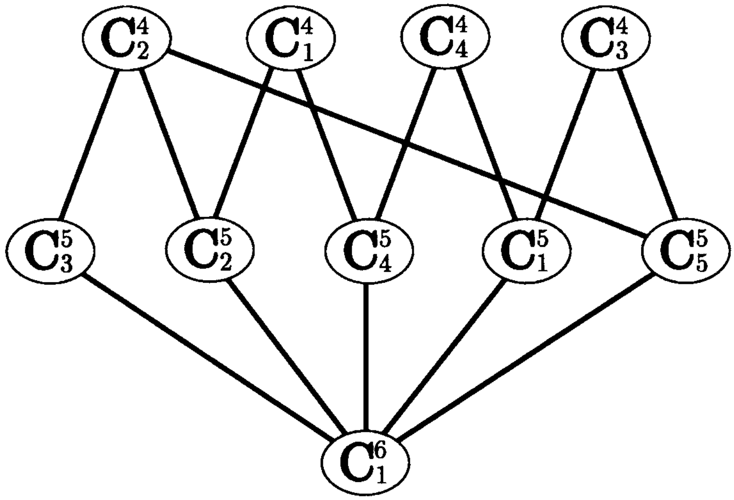

Figure 4. A Semilattice Constructed from Figure 3.

$$\mathbf{C}_1 \sqcap \mathbf{C}_2 \equiv \{C_1 \cap C_2 : C_1 \in \mathbf{C}_1, C_2 \in \mathbf{C}_2\} \setminus \{\emptyset\}.$$

For example

$$\mathbf{C}_1^4 \sqcap \mathbf{C}_3^4 = \mathbf{C}_1^5 .$$

It is easy to verify that $(\Sigma, \sqcap)$ is a semilattice (associative, commutative, idempotent, with $\mathbf{U}'$ as an absorption element).

The semilattice of clusterings for our example is presented in Figure 4.

3.2 Relations Without a Symmetric-Acyclic Decomposition

There are several possible approaches to networks that do not have an exact ranked clusters structure (and so are not ideal).

The symmetric-acyclic decomposition (S,Q) described above can be improved by setting

$$S = R \cap R^T \quad \text{and} \quad Q = R \setminus \overline{S},$$

where $\overline{S}$ replaces S in the definition of Q. In the ideal case this still produces the same asymmetric-acyclic decomposition; but in the nonideal case it removes all asymmetric inconsistencies from diagonal blocks.

3.2.1 Deleting Arcs

If there is no symmetric-acyclic decomposition, a useful procedure is arc deletion. The set

$$\Delta = \overline{S} \cap (R \setminus S)$$

gives us asymmetric arcs inside the otherwise symmetric classes of **U**. For Table 1, Δ identifies exactly the three asymmetric ties, the italicized 0's, within the diagonal blocks: $\Delta = \{(j,k),(o,n),(r,p)\}$. These ties need to be deleted or symmetrized. If the only violations of the symmetric-acyclic assumption concerned symmetry, the analysis would be complete. In the example, Q is not acyclic and we have to delete some arcs to make it acyclic. Here, the bold faced 1's above the diagonal would be located and deleted leaving an exact model (with the inconsistencies noted). In general, however, the identification of the smallest number of arcs to be deleted to obtain an acyclic model is not an easy problem.

3.2.2 Iteration of the Symmetric-Acyclic Decomposition

Also, we can iterate the above decomposition procedure until we obtain a graph without edges.

Let $\mathbf{U}_0 := \mathbf{U}$, $R_0 := R$ and set $i := 0$. Then repeat the following steps:

(1) Determine the symmetric part of R_i (the edges)

$$S_i := (R_i \cap R_i^T) \setminus I_{\mathbf{U}_i}$$

where $I_{\mathbf{U}_i} = \{(x,x) : x \in \mathbf{U}_i\}$ is the identity relation on $\mathbf{U}_i$.

(2) If no edge exists, $S_i = \varnothing$, stop iterating,

(3) Shrink the symmetric components, producing a new reduced graph $(\mathbf{U}_{i+1}, R_{i+1})$ determined by:

$$\mathbf{U}_{i+1} := \mathbf{U}_i / S_i^*$$

$$R_{i+1} := \{(X,Y) : \exists x \in X \; \exists y \in Y : xR_iy\}$$

(4) Increase the counter of steps $i := i + 1$.

The obtained clusterings are nested (and form a hierarchy) and all clusters are strongly connected. If the final clustering is not acyclic we add an additional step in which we factorize the graph according to strong connectivity. The final graph is the *condensation* of the original network (Harary, Norman, and Cartwright 1965, pp 57-65).

3.3 A Generalized Blockmodeling Approach

Blockmodeling methods were established in social network analysis to partition units in a social network in terms of the structural information contained in their network ties. Hummon and Carley (1993) showed that blockmodeling has been a major focus of activity for network analysts. Methods were invented, for example, CONCOR (Breiger, Boorman, and Arabie 1975) or adapted from extant clustering methods, for example STRUCTURE (Burt 1976). This initial work focused on structural equivalence (Lorrain and White 1971) where units are structurally equivalent if they are connected to the rest of the network in identical ways. Subsequent work, for example Sailer (1978/1979) and White and Reitz (1983), generalized this concept to regular equivalence where two units are regularly equivalent if they are equivalently connected to equivalent others. They have the same types of neighbors. Further work dealt with automorphic equivalence (see, for example, Borgatti and Everett 1992; Pattison 1988).

The partition of the units of a network into clusters generates a blockmodel where the (new) units are these clusters, called positions[4] The set of ties between two positions forms a block. Those ties in a block (i.e. between all units in the two positions) are used to construct — and thereby summarize — the ties between positions. In our discussion of the symmetric-acyclic decomposition, we described an ideal block pattern but said nothing of how the blocks would be represented in a blockmodel. The *image* of a blockmodel is a matrix or pictorial representation of the ties between positions. In essence, Figure 1 is such an image.

An appropriate generalization of the equivalence idea is one where each block, of a particular partition, is free to conform to a different pattern. This led Batagelj (1997) and Doreian, Batagelj, and Ferligoj (1994) to the definition of several types of connections inside and between the clusters as different types of blocks. Some of them are presented in Table 2. From the definition of structural equivalence it follows that there are two basic blocks: null and complete (Batagelj, Ferligoj, and Doreian 1992). Batagelj, Doreian, and Ferligoj (1992) proved that regular equivalence produces two types of blocks: null and regular (see Table 2).

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4. We have been using blockmodeling terms in discussing the symmetric-acyclic decomposition. Position is used here in the same sense as used in Section 2. Borgatti and Everett (1992) provide a lucid discussion of this concept.

Table 2. Characterizations of Types of Blocks

null	nul	all 0 (except may be diagonal)
complete	com	all 1 (except may be diagonal)
row-regular	rre	each row is 1-covered
col-regular	cre	each column is 1 -covered
row-dominant	rdo	∃ all 1 row (except may be diagonal)
col-dominant	cdo	∃ all 1 column (except may be diagonal)
regular	reg	1-covered rows and 1-covered columns
non-null	one	∃ at least one 1

Another block type, introduced here, is necessary for the symmetric-acyclic decomposition of networks. This is the symmetric block. A block is *symmetric* if

$$\forall\,(x,y) \in C_i \times C_j : (xRy \Leftrightarrow yRx)\,.$$

Note that for nondiagonal blocks this condition involves a pair of blocks $R(C_i,C_j)$ and $R(C_j,C_i)$.

3.3.1 Formalization of Blockmodeling

The point of departure is, as before, a network with a set of units, $\mathbf{U}$, and a relation $R \subseteq \mathbf{U} \times \mathbf{U}$. Let $\mathbf{Z}$ be a set of positions or images of clusters of units. Let $\mu : \mathbf{U} \to \mathbf{Z}$ denote a mapping which maps each unit to its position. The cluster of units $C(t)$ with the same position $t \in \mathbf{Z}$ is

$$C(t) = \mu^{-1}(t) = \{x \in \mathbf{U} : \mu(x) = t\}\,.$$

Therefore

$$\mathbf{C}(\mu) = \{C(t) : t \in \mathbf{Z}\}$$

is a partition (clustering) of the set of units $\mathbf{U}$.

A *(generalized) blockmodel* is an ordered quadruple $\mathbf{M} = (\mathbf{Z}, K, \mathbf{T}, \pi)$ where: (a) $\mathbf{Z}$ is a set of positions; (b) $K \subseteq \mathbf{Z} \times \mathbf{Z}$ is a set of *connections* between positions; (c) $\mathbf{T}$ is a set of predicates used to describe the types of connections between clusters in a network; we assume that nul $\in \mathbf{T}$; and (d) A mapping $\pi : K \to \mathbf{T} \setminus \{\text{nul}\}$ assigns predicates to connections.

A (surjective) mapping $\mu : \mathbf{U} \to \mathbf{Z}$ determines a blockmodel, $\mathbf{M}$, of a network $\mathbf{N}$ iff it satisfies the conditions:

$$\forall\,(t,w) \in K : \pi(t,w)(C(t),C(w))$$

and

$$\forall\, (t,w) \in \mathbf{Z} \times \mathbf{Z} \setminus K : \text{nul}\,(C(t),C(w))\,.$$

Let $\approx$ be an equivalence relation over $\mathbf{U}$. It partitions the set of units $\mathbf{U}$ into clusters

$$[x] = \{y \in \mathbf{U} : x \approx y\}\,.$$

We say that $\approx$ is *compatible* with $\mathbf{T}$ or is a $\mathbf{T}$-equivalence over a network $\mathbf{N}$ iff

$$\forall\, x,y \in \mathbf{U},\ \exists T \in \mathbf{T} : T([x],[y])$$

It is easy to verify that the notion of compatibility for $\mathbf{T} = \{\text{nul},\text{reg}\}$ reduces to the usual definition of regular equivalence. Similarly, compatibility for $\mathbf{T} = \{\text{nul},\text{com}\}$ reduces to structural equivalence.

For a compatible equivalence $\approx$ the mapping $\mu : x \to [x]$ determines a blockmodel with $\mathbf{Z} = \mathbf{U}/\approx$.

3.3.2 Optimization

The problem of establishing a partition of units in a network in terms of a selected type of equivalence is a special case of a *clustering problem* that can be formulated as an optimization problem: determine the clustering $\mathbf{C}^*$ for which

$$P(\mathbf{C}^*) = \min_{\mathbf{C} \in \Phi} P(\mathbf{C})$$

where $\mathbf{C}$ is a clustering of a given *set of units* $\mathbf{U}$, Φ is the set of all feasible clusterings and $P : \Phi \to \mathbb{R}_0^+$ the *criterion function.*

Criterion function $P(\mathbf{C})$ has to be *sensitive* to the selected type of equivalence:

$$P(\mathbf{C}) = 0 \Leftrightarrow \mathbf{C} \text{ determines an equivalence of the selected type.}$$

3.3.3 Criterion Functions

One of the possible ways of constructing a criterion function that directly reflects the selected type of equivalence is to measure the fit of a clustering to an ideal one with perfect relations within each cluster and between clusters according to the selected type of equivalence.

Given a clustering $\mathbf{C} = \{C_1, C_2, \ldots, C_k\}$, let $\mathbf{B}(C_u,C_v)$ denote the set of all ideal blocks corresponding to block $R(C_u,C_v)$. Then the global inconsistency of clustering $\mathbf{C}$ can be expressed as a criterion function

$$P(\mathbf{C}) = \sum_{C_u, C_v \in \mathbf{C}} \min_{B \in \mathbf{B}(C_u,C_v)} d(R(C_u,C_v),B)$$

where the term $d(R(C_u,C_v),B)$ measures the difference (inconsistency) between the block $R(C_u,C_v)$ and the ideal block B. The function d should be *compatible* with the selected type of equivalence:

$$B_i \notin \mathbf{B}(C_u,C_v) \wedge B_j \in \mathbf{B}(C_u,C_v) \Rightarrow d(B_i,B_j) > 0$$

Given a set of types of connections $\mathbf{T}$ and a block $R(C_u,C_v)$, $C_u,C_v \subseteq \mathbf{U}$ can determine the strongest (according to the ordering of the set $\mathbf{T}$) type T which is satisfied the most by $R(C_u,C_v)$. In this case we set

$$\pi(\mu(C_u),\mu(C_v)) = T$$

The obtained optimization problem can be solved by local optimization, using, for example a relocation algorithm (Batagelj, Doreian, and Ferligoj 1992).

3.3.4 Pre-Specified Blockmodels

The pre-specified blockmodeling starts with a blockmodel specified, in terms of substance, *prior to an analysis*. Batagelj, Ferligoj, and Doreian (1998) presented methods where a set of observed relations are fitted to a pre-specified blockmodel. Given a network, a set of ideal blocks is selected, a reduced model is formulated, and partitions are established by minimizing the criterion function.

In the example of the next section, the pre-specified blockmodel is acyclic with symmetric diagonal blocks. The permitted blocks for such a model are null (denoted nul), symmetric (sym), and 'contains at least one 1' (one). For a clustering into 4 positions, a symmetric-acyclic type of models can be specified as:

sym	nul	nul	nul
nul,one	sym	nul	nul
nul,one	nul,one	sym	nul
nul,one	nul,one	nul,one	sym

In concrete analyses, sym on the diagonal, and its subtypes nul and com, usually are included.

In our discussion thus far, all inconsistencies are alike in the sense that they contribute equally to the total number of inconsistencies. Given the ranked clusters model, it is possible to argue that some inconsistencies are more consequential than others. Following this logic, asymmetric ties down from a higher to a lower level are the most serious. The computed criterion function can include having a higher penalty for these ties (e.g., 100). Asymmetric ties within clusters seem the next most important type of error. The modification to the criterion function can have a milder penalty for these ties (e.g., 10). Finally, inconsistencies in the blocks below the main diagonal can be specified as minor (e.g., 1)[5] Using these values, the penalty matrix is:

10	100	100	100
1	10	100	100
1	1	10	100
1	1	1	10

4. Applications

We consider two examples. One is the constructed network in Figure 2 and the other features the Ragusan noble families (Krivošic 1990).

4.1 The Constructed Example

If the inconsistencies with the ranked clusters model in the relation in Table 1 are not present, all methods delineate the true structure correctly. Also, if only the violations of the symmetry condition within positions are present, all methods are successful.

With all of the inconsistencies shown in Table 1 in the data, the blockmodeling optimization approach locates the clusters correctly and reports all of the 5 inconsistencies using the model specification and penalties described in the previous section. However, the symmetric-asymmetric decomposition does not fare as well. The reasons for this are evident in Figure 2 representing the network from Table 1. As noted above, the figure was constructed in terms of clusters with ties that were inconsistent with an ideal ranked clusters model. In the decomposition, the algorithm establishes in the first step $\{j,k,l\}$, $\{d,e\}$ and $\{m,n,o\}$ correctly as subsets internally connected through mutual ties and not connected in that way to any other units. The remaining three clusters are identified also as being linked internally through mutual ties. The presence of the mutual tie between a and g connects the cluster $\{a,b,c\}$ to cluster $\{f,g,h,i)\}$ and the mutual tie between i and q connects the second of these clusters to $\{p,q,r,s\}$. Therefore, all three clusters are connected via symmetric ties. When the symmetric components are shrunk, there

5. If only nul and one blocks are specified below the diagonal, there are no inconsistencies. Future work will explore the specification more stringent block types below the diagonal.

are only four units for the next iteration. As three of them have symmetric ties the algorithm ends after second iteration with only two levels. The first consists of cluster $\{j,k,l\}$ and the second one of all other units.

One methodological interpretation is that the optimization approach is more robust in the presence of violations of the acyclic condition while the symmetric-acyclic decomposition is not. The latter procedure assumes perfect acyclic data. In contrast, the blockmodeling optimization concedes the presence of some inconsistencies and reports both partitions and inconsistencies rather than being overly affected by the latter.

4.2 Ragusan Families Marriage Network

Krivošić (1990) presented some interesting data for the population of Ragusa (Dubrovnik), a republic for most of its history. He also constructed two matrices describing the marriage networks of the Ragusan noble families in the 16th century and in the 18th century (and the beginning of 19th century). The matrix of the marriage relation for the second period is presented in Table 3 where the rows represent the families of the groom and the columns represent the houses of the bride. While these data are similar to the well known Padgett marriage network for the elite Florentine families in the 15th century (Padgett and Ansell 1993), they are directed and valued. Men marrying women and women marrying men are distinguished. The relation in Table 3 is 'men marrying women' and has been organized in conformity with the ranked clusters model fitted to these data. While the source data were valued (with counts of marriages between families), they were dichomoized to record only the presence or absence of marriage ties between families.

Ragusa was settled in 7th century, as reported by Constantine Porphyrogenite, by fugitives from Epidaurum after its destruction. Ragusa was for a time under Byzantine protection, becoming a free commune as early as 12th century. Ragusa quickly grew into a free city-state. They prospered unhindered thanks primarily to their clever diplomacy and great skill in balancing the great powers, formally recognizing and paying tribute alternately to one then another. Napoleon, having destroyed the Venetian Republic in 1797, put an end to the Republic of Ragusa in 1806, which subsequently came under Austrian control until the fall of the Austro-Hungarian monarchy in 1918.

The Ragusan nobility evolved in the 12th century through the 14th century and was finally established by statute in 1332. After 1332, no new family was accepted until the large earthquake in 1667.

In Ragusa all political power was in the hands of male nobles older than 18 years. They were members of the Great Council (*Consilium majus*) which had the legislative function. Every year, 11 members of the Small Council (*Consilium minus*) were elected. Together with a duke — who was

Table 3. Permuted Ragusan Families Marriage Network, 18th and 19th Century

		14	20	1	3	7	10	12	19	22	23	2	8	9	6	13	11	16	21	4	5	15	17	18
Natali	14	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
Slatarich	20	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
Basilio	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	1	·	·	·	·	·
Bonda	3	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	2	·	·	·	·	·
Cerva	7	·	·	·	·	·	·	·	·	·	·	1	·	·	1	1	·	·	1	·	·	·	·	·
Gondola	10	·	·	·	·	·	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·
Gradi	12	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	1	1	3	·	·	·	·	·
Saraca	19	·	·	·	·	·	·	·	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·	·
Tudisi	22	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	1	·	·	·	·	·
Zamagna	23	·	·	·	·	·	·	·	·	·	·	1	·	2	·	·	·	·	1	·	·	·	·	·
Bona	2	·	·	·	·	1	·	·	·	·	·	·	·	·	·	2	2	·	1	·	·	·	·	·
Georgi	8	·	·	·	2	1	·	·	·	·	1	1	·	·	·	·	·	4	·	·	·	·	·	·
Ghetaldi	9	1	·	·	·	·	1	·	1	·	·	1	1	·	·	·	·	·	·	·	·	·	·	·
Caboga	6	1	·	·	·	1	·	·	1	·	·	·	·	·	·	1	·	·	1	·	·	·	·	·
Menze	13	·	·	·	·	·	·	·	·	·	1	·	·	1	1	·	·	·	·	·	·	·	·	·
Goze	11	·	·	1	1	·	·	2	·	1	·	2	·	·	·	·	2	·	2	·	·	·	·	·
Poza	16	·	·	·	2	·	·	·	·	·	·	·	·	·	·	·	1	1	1	·	·	·	·	·
Sorgo	21	·	1	·	1	·	·	·	·	·	1	2	·	·	·	·	1	·	1	·	·	·	·	·
Bosdari	4	·	·	·	·	·	·	·	·	·	1	1	·	·	·	·	·	·	·	·	·	·	·	·
Bucchia	5	·	·	·	·	·	·	·	·	·	·	·	·	·	·	1	·	·	·	·	·	·	·	·
Pauli	15	·	·	·	·	·	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·
Ragnina	17	·	·	·	1	·	·	·	·	·	·	1	·	·	·	·	1	·	1	·	·	·	·	·
Resti	18	·	·	·	·	·	·	1	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·

elected for a period of one month — it had both executive and representative functions. The main power was in the hands of the Senat (*Consilium rogatorum*) which had 45 members elected for one year.

This organization prevented any single family, unlike the Medici in Florence, from prevailing. Nevertheless the historians agree that the Sorgo family was always among the most influential families.[6] For example: (a) in the 17th century, 50% of dukes and senators were from the following five families — Bona, Gondola, Goze, Menze, and Sorgo; (b) in the 18th century, 56% of senators were from five families — Sorgo, Goze, Zamagna, Caboga, and Georgi; and (c) in the last 8 years of Republic, 50% of dukes were from five families: Sorgo, Goze, Gradis, Bona, and Ragnina.

A major problem facing the Ragusan noble families was that there were decreases of their numbers and no noble families were in the neighboring areas (which were under Turkish control). As a result, they became more and more closely related (1566 — *"quasi tutti siamo congionti in terzo et in quarto grado di consanguinita et affinita"*) — the marriages between relatives of only 3rd and 4th removed were frequent.

4.2.1 Earlier Analyses

Batagelj (1996a) found that the most influential families, according to the indices of centrality, in the second period (18th and 19th century) were Sorgo, Bona, and also Zamagna, Cerva and Menze. He obtained two basic clusters for this period by using the generalized blockmodeling approach:

Cluster 1: Basilio, Bona, Bonda, Caboga, Cerva, Georgi, Ghetaldi, Gondola, Goze, Gradi, Menze, Poza, Saraca, Sorgo, Tudisi, and Zamagna.

Cluster 2: Bosdari, Bucchia, Natali, Pauli, Ragnina, Resti, and Slatarich.

The second cluster contains all new families accepted after the earthquake.

The structure obtained by Batagelj is an example of a center-periphery model. Most marriages were among the families of the first cluster, there was no marriage among families of the second cluster, and there were only few marriages between the two clusters.

4.2.2 A Network Decomposition

In this section only the Ragusan families marriage network of the second period is analyzed. We remind readers that the direction of the asymmetric ties, in general, is arbitrary. Here, the permitted ties are 'down'.

Using the acyclic decomposition with respect to strong connectivity, three levels were obtained (see Figure 5). In the first and the third level only

6. Here we simply report their findings and note that the Goze family was seen as less influential.

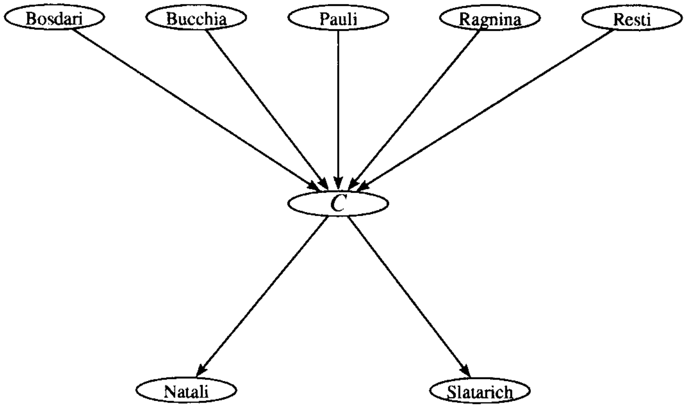

Figure 5. An Acyclic Model of Ragusan Families Network.

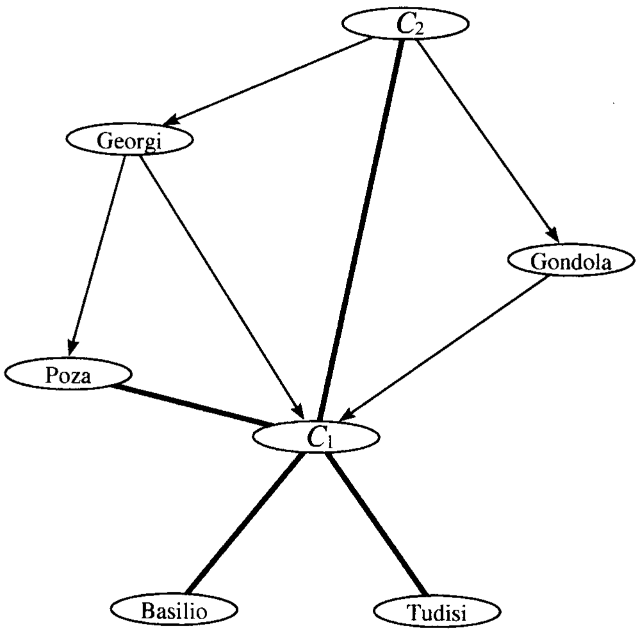

Figure 6. The Graph of Seven First Order Symmetric Components of the Middle Cluster.

Figure 7. The Graph of Three Second Order Symmetric Components of the Middle Cluster.

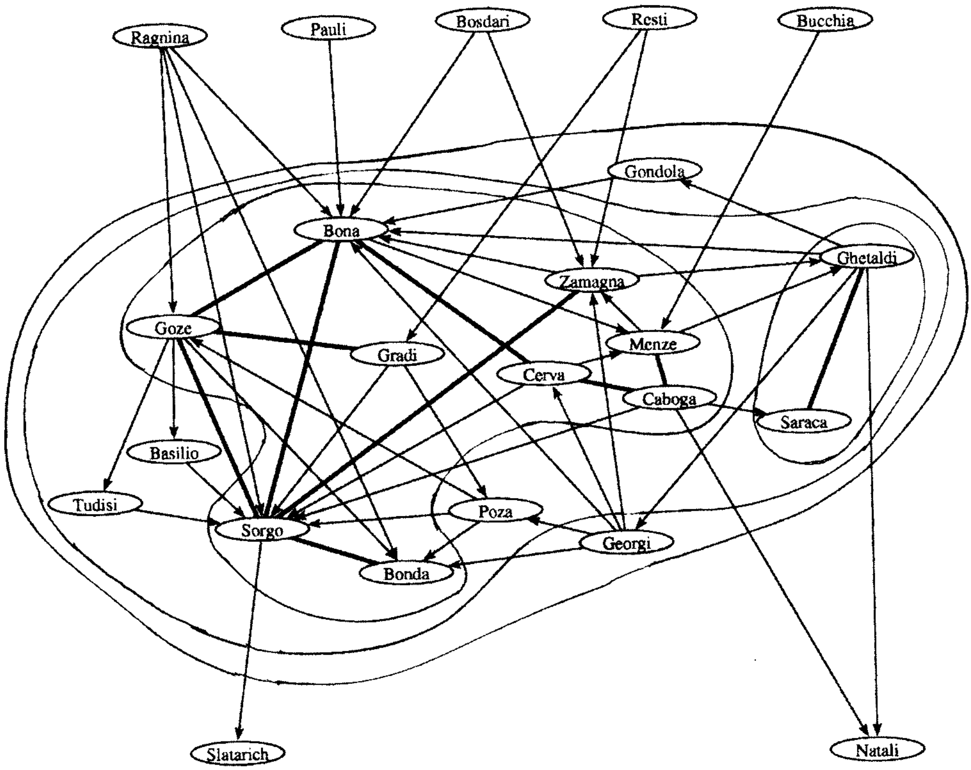

Figure 8. A Symmetric-Acyclic Decomposition of Ragusan the Families Network.

single families are present. The middle level consists of a cluster C with sixteen families. The men from the families of the first level choose wives from the families of the second level and the men from the second level choose wives from the families of the third level. It is worth noting that five families are *sources* in the network providing husbands to the large 'middle' cluster of families, among which husbands (and wives) circulate. These families are *transmitters* and there are two families that are *sinks*.

The internal structure of the second cluster (the middle level in Figure 5) can be revealed by the iteration of the symmetric-acyclic decomposition method. The symmetric components of the middle cluster C are:

$C_1 =$ {Bona, Bonda, Caboga, Cerva, Goze, Gradi, Menze, Sorgo, Zamagna}; $C_2 =$ {Ghetaldi, Saraca}; $C_3 =$ {Basilio}; $C_4 =$ {Georgi}; $C_5 =$ {Gondola}; $C_6 =$ {Poza}; and $C_7 =$ {Tudisi}.

The graph of the seven (first order) symmetric components (clusters) of the cluster C is presented in Figure 6. This reduced graph has three (second order) symmetric components: $C'_1 = \{C_1, C_2, C_3, C_6, C_7\}$; $C'_2 =$ {Georgi} and C'_3 = {Gondola}.

Their reduced graph (see Figure 7) is a single symmetric component. All three steps of the hierarchical decomposition can be seen in Figure 8.

4.2.3 A Generalized Blockmodeling Approach

Based on a three levels network decomposition two pre-specified models can be assumed. (See Table 2 for the block types.) The first is

nul,reg,sym	nul	nul
nul,cre,rre	nul,reg,sym	nul
nul,cre,rre	nul,cre,rre	nul,reg,sym

The second pre-specified model is similar but with one difference: 'reg' is excluded from the diagonal elements (this model has more precisely defined blocks in the lower triangle than in the general case from Section 3.3.4):

nul,sym	nul	nul
nul,cre,rre	nul,sym	nul
nul,cre,rre	nul,cre,rre	nul,sym

Only the second model is symmetric-acyclic. The penalties in each cell were assumed in accordance with Section 3.3.4. For both models the same clustering of families was obtained (compare Figure 8):

Cluster 1: Natali, Slatarich.

Cluster 2: Basilio, Bona, Bonda, Caboga, Cerva, Georgi, Ghetaldi, Gondola, Goze, Gradi, Menze, Poza, Saraca, Sorgo, Tudisi, Zamagna.

Cluster 3: Bosdari, Bucchia, Pauli, Ragnina, Resti.

The permuted original relational matrix for this clustering was given in Table 3. The obtained models are

nul	nul	nul
cre	reg	nul
nul	rre	nul

and

nul	nul	nul
cre	sym	nul
nul	rre	nul

The first has no inconsistencies and, for the second, there are 28 inconsistencies in the middle diagonal block. These inconsistencies are produced by non-symmetric ties.

For the clusterings with larger number of clusters (4, 5 or 6) the solutions for the first model have no inconsistencies. All obtained clusterings have the same second cluster. There are some splits, as we can expect from the theory of decomposition, of the first and/or the third cluster.

When the structure inside the cluster in one of the levels of the symmetric-acyclic decomposition is not clearly symmetric some other types of the structure can be tested by applying the blockmodeling approach to this cluster separately. In this case an appropriate model should be pre-specified. As the iterative decomposition is based on the symmetric components of the cluster the blockmodeling approach is more appropriate to search for also other types of structures (for example, a center-periphery structure).

Using pre-specified blockmodeling on the subgraph induced by the cluster *C*, the following results were obtained which imply the ordering within the second cluster in Table 3 is:

(1) **symmetric clusters model** ({nul} on out-diagonal and {com,sym} on the diagonal with penalties 1) gives for 7 clusters the same clustering as was obtained at the first step of the iterative procedure;

(2) **center-periphery model** ({nul,one} with penalty 1 on the out-diagonal, {com} with penalty 10 on the diagonal, {nul} with penalty 100 as the first diagonal element) gives the following 4 clusters of *C*:

Periphery: Basilio, Bonda, Cerva, Gondola, Gradi, Saraca, Tudisi, Zamagna.

Center 1: Bona, Georgi, Ghetaldi.

Center 2: Caboga, Menze.

Center 3: Goze, Poza, Sorgo.

5. Discussion

The above analyses can be located within the domain of blockmodeling insofar as they were designed to partition social networks. However, several methodological conclusions can be drawn from these analyses and are worth stressing. On the blockmodeling side, we emphasize that the analyses featured a generalized version of blockmodeling and pre-specified models were used. Both represent major departures from conventional blockmodeling. The symmetric-acyclic decomposition facilitates (or forces) analysts to consider the internal structure of the blocks of a blockmodel. This is a desirable feature that is absent in most blockmodeling efforts. Of course, once partitions and the induced blocks have been discerned via blockmodeling, their internal structure can be examined. Our point is this is seldom done and the proposed decomposition naturally directs our attention to the internal structure of blocks.

A second feature worth noting is that the two methods proposed here are concerned with hierarchy and levels in some sense. As is made clear in the multiple clusterings in the constructed example, there need not be a single partition that fits better than all others. This complicates the notion of hierarchy and more care may be in order when hierarchy is discussed. The number of levels is obtained from the Hasse diagram (for example, in Figure 3) and seems a good foundation on which to base discussions of levels and, more generally, hierarchy. Conventional blockmodeling can lead to structures that can be interpreted in term of hierarchy. However, as implemented here the methods we propose are predicated on hierarchy and the best fitting hierarchical partitions can be delineated. This is seen in the comparison of the partition reported by Batagelj (1996a) and summarized in Section 4.2.1 here, with the partition shown in Section 4.2.2. One is richer than the other through capturing hierarchy.

A third feature to note is that, both the symmetric-acyclic decomposition and the ranked clusters blockmodel methods have their origins located firmly in substance. By thinking in terms of ranked clusters, as a theoretical specification, we were able to formulate the new block type, a new blockmodel type and the symmetric-acyclic decomposition as tools for delineating ranked structures.

However, some problems do remain and concern the fact that the ranked clusters model did not fit exactly. By focusing our attention on the block types we were able to locate asymmetric ties within diagonal blocks and those ties that violate the acyclic specification. The blockmodeling approach also enabled us to weight each type of inconsistency differently. The relative weighting of the types of inconsistencies that we used seems appropriate: inconsistencies of the acyclic condition seem more serious. We have not explored the use of differing magnitudes of the weights beyond the one described in Section 3.3.4.

The methods we use allows us to describe the best fitting partitions of a given type and are appropriate even when the ranked clusters model does not fit the network data perfectly. No statistical measure of fit is needed beyond this: it is hard to beat the best as the criterion function counts only the exceptions to a perfect ranked clusters structure. Beyond this, however, some additional thought needs to be given to the use of measures of fit. While we report measures of fit, we do not have an assurance that the structure we have described as a ranked clusters model really is a ranked clusters model. Ideally, we need some threshold so that a values below this threshold are consistent with a ranked clusters structures and values above it are not. Sufficiently large values of the criterion funcion mean there are too many inconsistencies to claim that the model is close enough to the ideal ranked clusters forms. Intuitively, the models we have developed in Section 4 seem to be ranked clusters models. After all, the hypothetical model was constructed in terms of ranked clusters with some inconsistencies built into the data and the generalized blockmodeling method located these inconsistencies. The movement of grooms or brides across the Ragusan noble families has a compelling structure and interpretation. Finally, we note that the symmetric-acyclic approach is very efficient for very large networks but is sensitive to the violations of the acyclic assumption of the models.

All computations were done by using the programs Pajek (for the symmetric-acyclic decomposition) and MODEL2 (for the generalized blockmodeling). Both are freely available for noncommercial use at the address: http://vlado.fmf.uni-lj.si/pub/networks.

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