An Optimizational Approach to Regular Equivalence

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An optimizational approach to regular equivalence *

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Most partitions of social network (relational) data are presented and interpreted with little attention to assessing the adequacy of the partitions. We present a new criterion function that operationalizes the concept of regular equivalence and provides a measure of the departure of any structure from an exact regular partition. We then use a local optimization procedure to locate partitions where the algorithm minimizes the criterion function. After applying these methods for constructed and empirical data sets, we compare our results with those obtained from using other methods.

As proposed by Lorrain and White (1971), and operationalized by Breiger *et al.* (1975), or by Burt (1976), actors are structurally equivalent if their links to the rest of the network are identical. Attempts to generalize this formulation date back at least to Sailer (1978) and have taken various forms. Integral to all formulations is the idea that actors are equivalent if they link in equivalent ways to other actors that are also equivalent. Regular equivalence as defined by White and Reitz (1983) is one such generalization and is the focus of this paper.

Most attempts to generate partitions of nodes in a social network do so by specifying a measure that purports to capture the extent to which each pair of nodes are equivalent and then invoking some clustering procedure for the matrix of similarities. Conceptually the

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two steps are divorced from each other: any measure of similarity can be used and any clustering procedure, in principle, can be used. Of course, there is room for debate over the relative merits of measures and algorithms, but, in general, the two steps are linked only by sequencing. Once a clustering has been obtained it can be interpreted in terms of the underlying network structure, but true assessment of the adequacy of the partition is seldom undertaken.

We propose an integrated approach to partitioning – or blocking – a social network in terms of regular equivalence. First, we construct an objective function directly in terms of regular equivalence and then use an optimization procedure to minimize the objective function. In this fashion, the measurement of equivalence and the clustering procedure are directly coupled and, in addition to providing a partition, the procedure provides a measure of the extent to which the derived partition departs from an ideal regularly partitioned structure.

The paper is structured as follows. In section 1 we lay out the basic ideas and show that if we are able to construct an objective function then the blocking problem can be expressed as an optimization problem. Section 2 contains a discussion of regular equivalence. As part of this, proposition 1 provides the conceptual foundation for constructing an appropriate criterion function. Section 3 lays out the criterion function and introduces a local optimization procedure for constructing a partition of the network in terms of regular equivalence. Section 4 provides a set of constructed examples and an empirical example taken from the literature. The paper finishes with a brief conclusion.

1. Basic notions

Let $E = \{X_1, X_2, \ldots, X_n\}$ be a finite set of *units*. In general the units are related by binary *relations*

$$R_t \subseteq E \times E, \qquad t = 1, \ldots, r$$

which determine a *network*

$$\mathcal{N} = (E, R_1, R_2, \ldots, R_r).$$

In order to focus on the establishment of a criterion function and the use of a local optimization procedure, we restrict our discussion to a single relation R described by a corresponding binary matrix $\boldsymbol{R} = [r_{ij}]_{n \times n}$

$$r_{ij} = \begin{cases} 1 & X_i R X_j \\ 0 & \text{otherwise} \end{cases}$$

In some applications, the r_{ij} can be nonnegative real numbers expressing the strength of the relation R between units X_i and X_j.

The main goal of blockmodelling is to identify, in a given network, *clusters* (classes) of units which play the same, or a similar, role – that is, they have the same or a similar connection pattern to other units. They form a *clustering*

$$\mathscr{C} = \{C_1, C_2, \ldots C_k\}$$

which is a partition of the set E:

$$\bigcup_i C_i = E$$

$$i \neq j \Rightarrow C_i \cap C_j = \emptyset$$

Each partition determines an equivalence relation (and vice versa). We use $\sim$ to denote the relation determined by partition $\mathscr{C}$. A *blockmodel* consists of structures obtained by identifying all units from the same cluster of the clustering $\mathscr{C}$. The problem of determining the classes of equivalent units is called the *blocking problem*.

Blockmodelling, as an empirical procedure, hinges on the idea that actors in a network can be grouped (into blocks) according to the extent to which they are equivalent, according to some *meaningful* definition of equivalence. In general, and without surprise, different definitions of equivalence lead to distinct partitions. Regardless of the definition there are two basic approaches to the equivalence of units in a given network (compare Faust 1988):

- the equivalent units have the same connection pattern to the *same* neighbors;

- the equivalent units have the same or similar connection pattern to (possibly) different neighbors.

The first type of equivalence is formalized by the notion of structural equivalence; the second by the notion of regular equivalence. We consider regular equivalence here. For a parallel treatment of structural equivalence see Batagelj *et al.* (1992).

Assume that we have a single relation network $\mathcal{N} = (E, R)$. Let Θ denote the set of all equivalence relations of a selected type (for example, regular or structural equivalences) over $\mathcal{N}$. Every equivalence relation $\sim$ on E determines a partition $\mathscr{C}$ of E, and vice versa. Let Φ denote the set of all partitions corresponding to the relations from Θ.

If we are able to construct a criterion function $P(\mathscr{C})$ with the properties:

P1. $P(\mathscr{C}) \geq 0$
P2. $p(\mathscr{C}) = 0 \Leftrightarrow \sim \in \Theta$,

then we can express the blocking problem as an optimization problem:

Determine the clustering $\mathscr{C}^* \in \Phi$ such that

$$P(\mathscr{C}^*) = \min_{\mathscr{C} \in \Phi} P(\mathscr{C})$$

If there are exact equivalences, then (by P2) the minimal value of $P(\mathscr{C})$ is 0. In the case when Θ is empty the optimization approach gives the solution(s) which differ(s) least from some ideal case.

Given a clustering $\mathscr{C} = \{C_1, C_2, \ldots, C_k\}$, let $\mathscr{B}(C_u, C_v)$ denote the set of all ideal blocks corresponding to block $R(C_u, C_v)$. Then the global error of clustering $\mathscr{C}$ can be expressed as

$$P(\mathscr{C}) = \sum_{C_u, C_v \in \mathscr{C}} \min_{B \in \mathscr{B}(C_u, C_v)} d(R(C_u, C_v), B)$$

where the term $d(R(C_u, C_v), B)$ measures the difference (error) between the block $R(C_u, C_v)$ and the ideal block B. The function d has to be compatible with the selected type of equivalence.

In the following we construct the function d for regular equivalence.

2. Regular equivalence

The equivalence relation $\cong$ on E is a regular equivalence on network $\mathcal{N} = (E, R)$ iff for all $X, Y, Z \in E$, $X \cong Y$ implies both

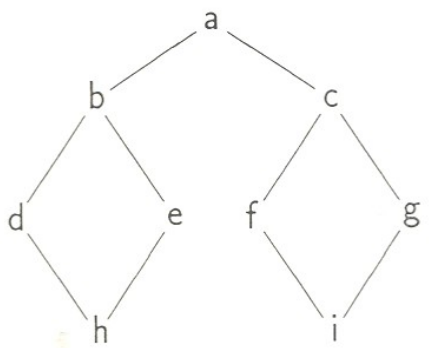
R1. $XRZ \Rightarrow \exists W \in E: (YRW \wedge W \cong Z)$
R2. $ZRX \Rightarrow \exists W \in E: (WRY \wedge W \cong Z)$

The construction of the criterion function for regular equivalence is based on the following result.

Proposition 1. Let $\mathscr{C} = \{C_i\}$ be a clustering corresponding to a regular equivalence $\cong$ on the network $\mathcal{N} = (E, R)$. Then each block $R(C_u, C_v)$ is either null or it has the property that there is at least 1 in each of its rows and in each of its columns.

Conversely, if for a given clustering $\mathscr{C}$ each block has this property then the corresponding equivalence relation is a regular equivalence.

The proof of this proposition is given in the appendix. Figure 1 contains a structure that illustrates this proposition. Actors a through i can be partitioned into 4 blocks K through N based on the definition of regular equivalence. Consider blocks L and M. The ties between these blocks are such that for each row for M sending ties to L there is a link (d sending to b, e sending to b, b sending to c, and g sending to c) and each column of L receiving ties from M has a link. No actor in block N sends ties to block K or L. Each block in the



		K	L		M				N	
		a	b	c	d	e	f	g	h	i
K	a		1	1						
L	b	1			1	1				
	c	1					1	1		
M	d		1						1	
	e		1						1	
	f			1						1
	g			1						1
N	h				1	1				
	i						1	1		

Fig. 1. Illustration of Proposition 1.

blockmodel has ties fitting the pattern described in the proposition and the whole structure has a nontrivial exact regular partition.

3. Optimization and regular equivalence

3.1. A criterion for regular equivalence

From the proposition it follows that regular equivalence produces two types of blocks:

- *null blocks* which have all entries 0; and
- *1-covered blocks*, which have in each row and in each column at least one 1.

Therefore we can use, as a measure of regularity of a block, the quantity

$$d(R(C_u, C_v), B) = \begin{cases} \#\text{ of 1-covered rows/columns} & B \text{ is null block} \\ \#\text{ of 0 rows/columns} & B \text{ is 1-covered block} \end{cases}$$

From the proposition it follows that the defined criterion function $P(\mathscr{C})$ is sensitive to regular equivalence:

$P(\mathscr{C}) = 0 \Leftrightarrow \mathscr{C}$ defines regular equivalence.

3.2. Algorithm

Often for a given optimization problem (Φ, P) there exist rules which relate each element of the set of feasible solutions Φ to some elements of Φ. We call them *local transformations*. They are the basis of the local optimization procedure which, starting with an element of Φ, repeats moving an element, determined by a local transformation, to reach a lower value of the criterion function.

The elements which can be obtained by local transformations from a given element are called neighbors – local transformations determine the *neighborhood relation* $S \subseteq \Phi \times \Phi$ in the set Φ. The *neighborhood* of element $X \in \Phi$ is called the set $S(X) = \{Y\colon XSY\}$. The

element $X \in \Phi$ is a *local minimum* for the neighborhood structure (Φ, S) iff

$$\forall Y \in S(X)\colon P(X) \leq P(Y)$$

The basic scheme of local optimization procedure is therefore very simple (see Foulds 1984):

determine the initial element $X_0 \in \Phi$, $X := X_0$;
while $\exists Y \in S(X)$: P(Y) $< P(X)$ **repeat** $X := Y$

To obtain a "good" solution and an impression about its quality we repeat the procedure with different (random) partitions X_0.

Although the basic scheme of local optimization procedure is very simple it is necessary to make some important decisions in its design. To obtain an efficient algorithm we have to consider each of the following:

- rich neighborhood structures increase the likelihood of reaching a global minimum. The extreme case represents the case in which all units are neighbors;
- rich neighborhood structures increase the time spent by each step of the algorithm;
- an efficient algorithm for generating neighbors should exist.

Usually the neighborhood relation in local optimization procedures over clusterings (partitions) is determined by the following two transformations:

- clustering $\mathscr{C}'$ is obtained from the clustering $\mathscr{C}$ by *moving* a unit X_k from cluster C_p to cluster C_q (*transition*):

$$\mathscr{C}' = \left(\mathscr{C} \setminus \{C_p, C_q\}\right) \cup \left\{C_p \setminus \{X_k\}, C_q \cup \{X_k\}\right\}$$

- clustering $\mathscr{C}'$ is obtained from the clustering $\mathscr{C}$ by *interchanging* units X_u and X_v from different clusters C_p and C_q (*transposition*):

$$\mathscr{C}' = \left(\mathscr{C} \setminus \{C_p, C_q\}\right) \cup \left\{\left(C_p \setminus \{X_u\}\right) \cup \{X_v\}, \left(C_q \setminus \{X_v\}\right) \cup \{X_u\}\right\}$$

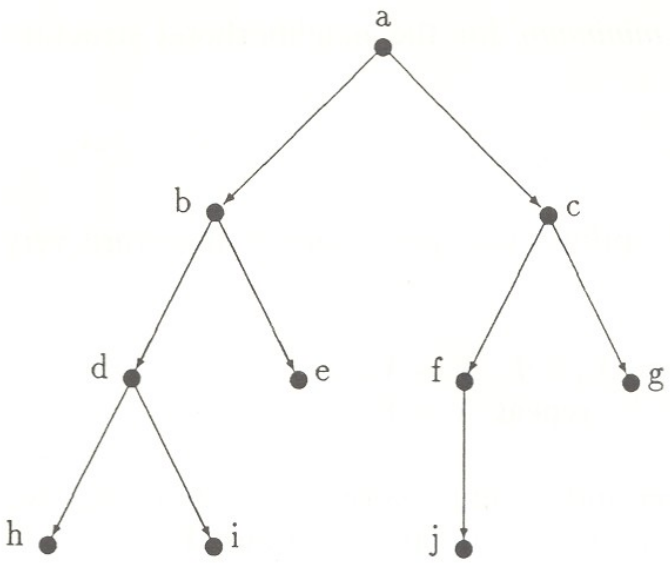


Fig. 2. Small tree.

In both cases only two clusters are changed. Therefore it is useful to introduce the quantity

$$\Delta P(\mathscr{C}, \mathscr{C}') = P(\mathscr{C}) - P(\mathscr{C}')$$

which allows faster tests of the condition $P(\mathscr{C}') < P(\mathscr{C})$.

4. Examples

For a graph there is, in general, a nonempty set of regular partitions (Borgatti and Everett 1989). For small graphs ($n \leq 12$) it is possible to obtain all regular partitions (which form a lattice) via an exhaustive search. We can locate all of the partitions generated by the following local optimization procedure in the corresponding lattices.

Figure 2 contains a directed graph for 10 nodes. By means of an exhaustive search, all exact regular equivalences in the graph can be found. Table 1 shows the distribution of the number of clusters in a partition and the existence of a regular partition. Attempts to cluster the nodes into 2, 3, 4, 6, 7, or 8 clusters yield no regular partitions. The third column of Table 1 lists the minimum value of the criterion function obtained by searching over all possible clusterings. An effort to split the graph into 5 clusters will yield one regular partition:

Table 1
Number of clusters and regular equivalence: small tree

Number of clusters	Regular partition	Minimum value criterion function
1	NO	6
2	NO	4
3	NO	6
4	NO	2
5	YES	0
6	NO	1
7	NO	2
8	NO	1
9	YES	0
10	YES	0

$\{\{a\}, \{b, c\}, \{d, f\}, \{e, g\}, \{h, i, j\}\}$. There is also one regular partition into 9 clusters: nodes h and i are clustered together, with the remaining nodes all singletons, and, of course, the trivial partition with only singletons. When the local optimization approach, in terms of regular equivalence, is used it finds both nontrivial regular parti-

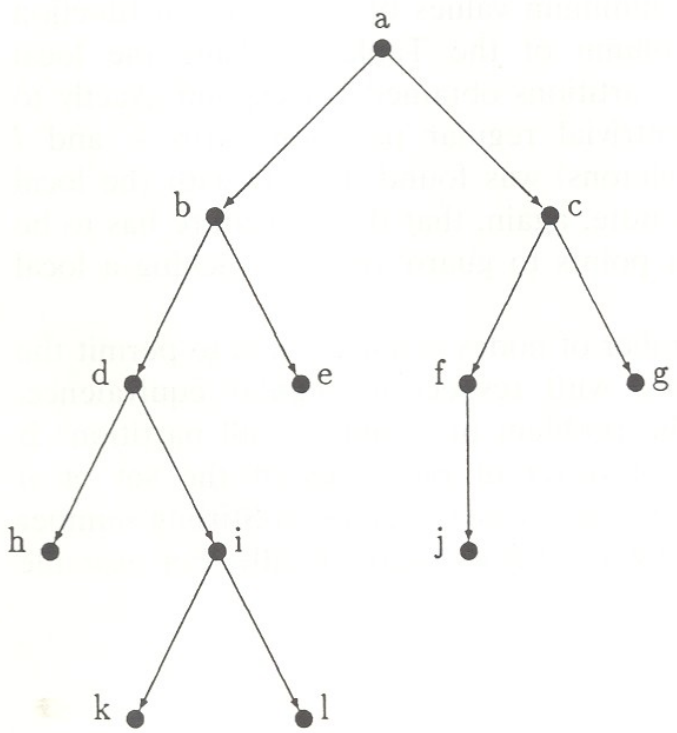


Fig. 3. Large tree.

Table 2
Number of clusters and regular equivalence: large tree

Number of clusters	Regular partition	Minimum value criterion function
1	NO	7
2	NO	4
3	NO	6
4	NO	4
5	NO	3
6	NO	2
7	NO	1
8	NO	1
9	NO	1
10	NO	1
11	YES	0
12	YES	0

tions. For all other number of clusters if finds partitions for which the obtained value of the criterion function are exactly as in Table 1.

The directed graph of Figure 3 is even more stark. Table 2 summarizes the attempts to cluster the nodes into specified number of clusters. If k is the number of clusters then for $1 \leq k \leq 10$ there are no regular partitions. The minimum values of the criterion function are shown in the third column of the Table 2. Using the local optimization approach, the partitions obtained correspond exactly to these values. The only nontrivial regular partition (with k and l clustered together plus singletons) was found directly with the local optimization algorithm. We note, again, that the procedure has to be repeated with random start points to guard against reaching a local optimal partition.

For these graphs, the number of nodes is low enough to permit the examination of all partitions with respect to regular equivalence. However, beyond $n = 12$ the problem of examining all partitions is intractable – the number of different partitions of the set of n elements into k nonempty disjoint subsets equals to Stirling number (of the second kind) $S(n, k)$ which grows exponentially. For example

$S(13, 6) = 9321312$
$S(15, 6) = 420693273$
$S(20, 8) = 15170932662679$

Therefore the complete enumeration for $n > 12$ is not feasible.

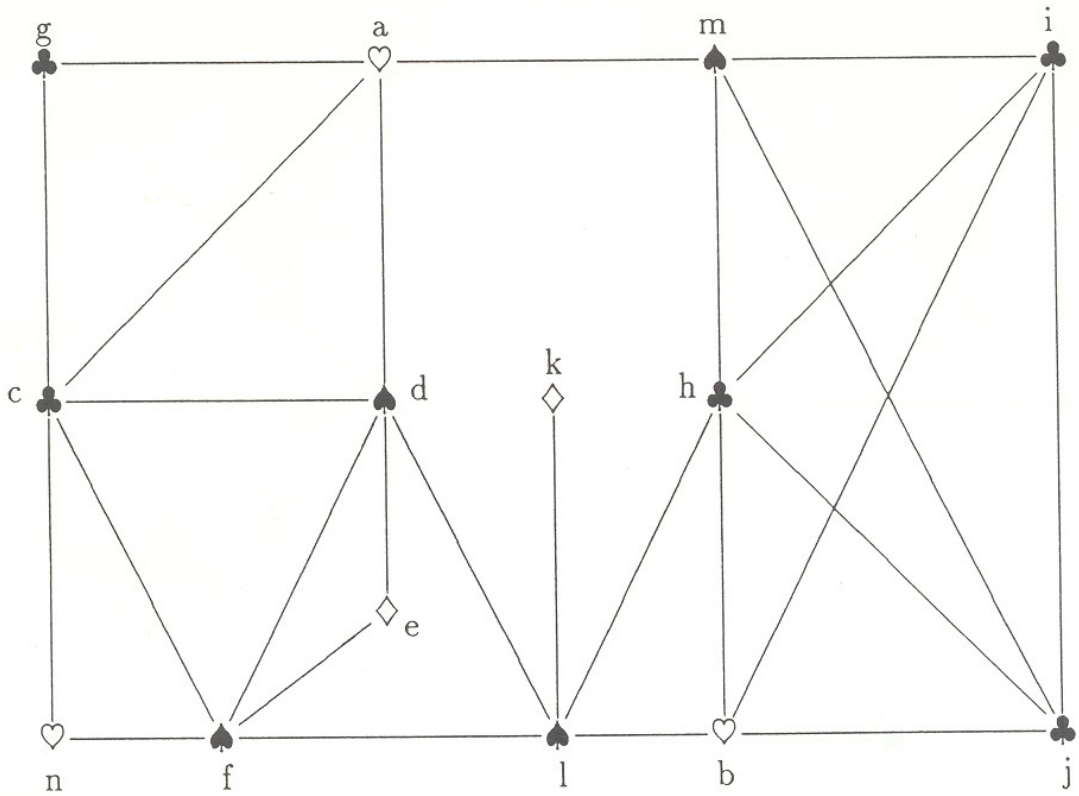


Fig. 4. Political actors – the first near regular partition ($k = 4$).

For the political actor network (Doreian 1988), there are 14 nodes, too many for a full exhaustive search. As the local optimization procedure is practical and, given its success on smaller graphs, we use it to locate regular positions. For small numbers of clusters, we can still validate our clustering via exhaustive search. For partitions with two clusters we obtained 12 regular partitions.

There are no regular partitions for $k = 3$ or for $k = 4$. When $k = 3$, the minimum value of the criterion function is 1 and when $k = 4$ the minimum value of the criterion function is 2. Doreian (1988: 273) describes a partition of the 14 political actors into 4 blocks as regular. This is shown in Figure 7. Before examining that partition, we examine the partitions (Figures 4 to 6) obtained via the local optimization when the criterion takes its minimum value of 2. The partition of Figure 4 is $\{a, b, n\}$, $\{e, k\}$, $\{d, f, l, m\}$ and $\{c, g, i, j, h\}$. The partition of Figure 5 differs only by the transposition of h and m between the last two clusters. The partition of Figure 6 is $\{a, c, i, j, m\}$, $\{e, k\}$, $\{b, d, f, l\}$ and $\{g, h, n\}$. When the criterion function is constructed for the partition of Figure 7, its value is 20. Clearly, the reported

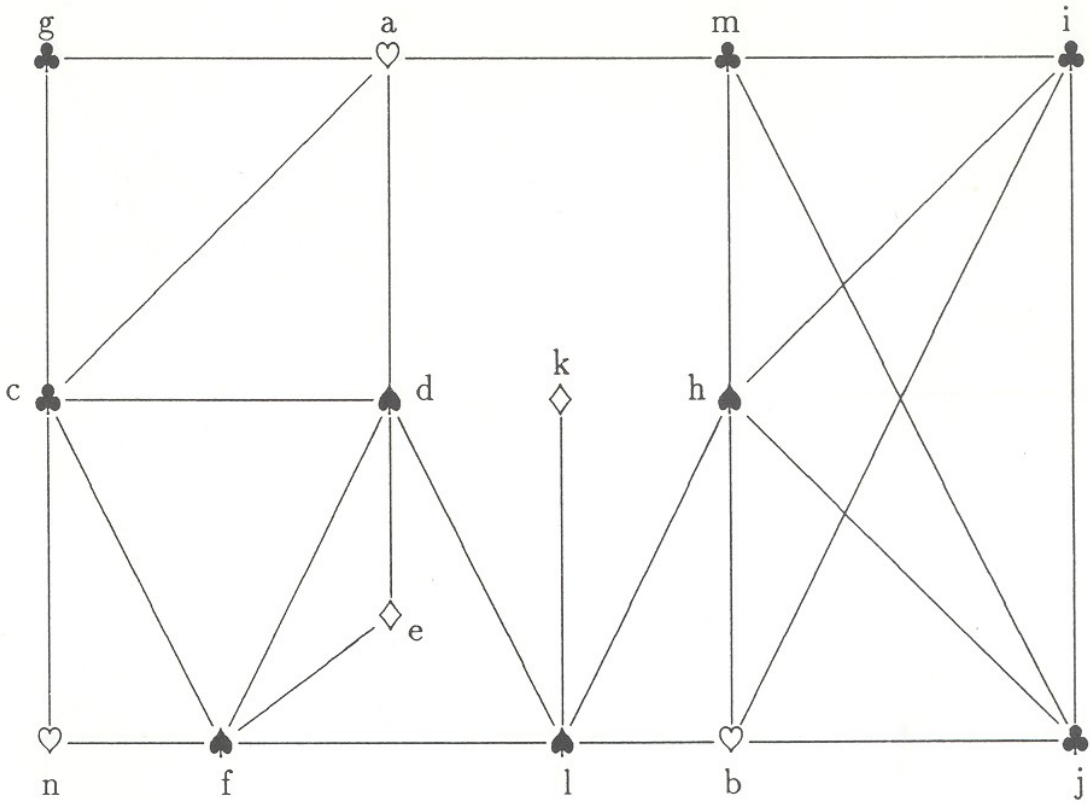


Fig. 5. Political actors – the second near regular partition ($k = 4$).

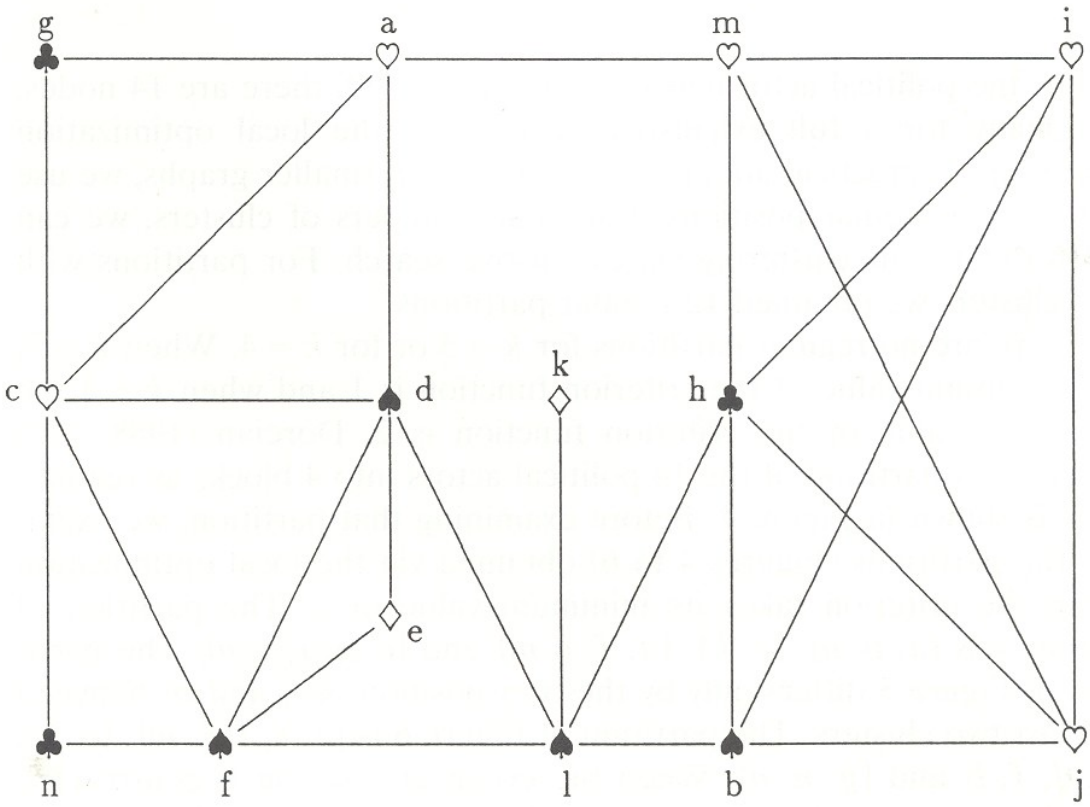


Fig. 6. Political actors – the third near regular partition ($k = 4$).

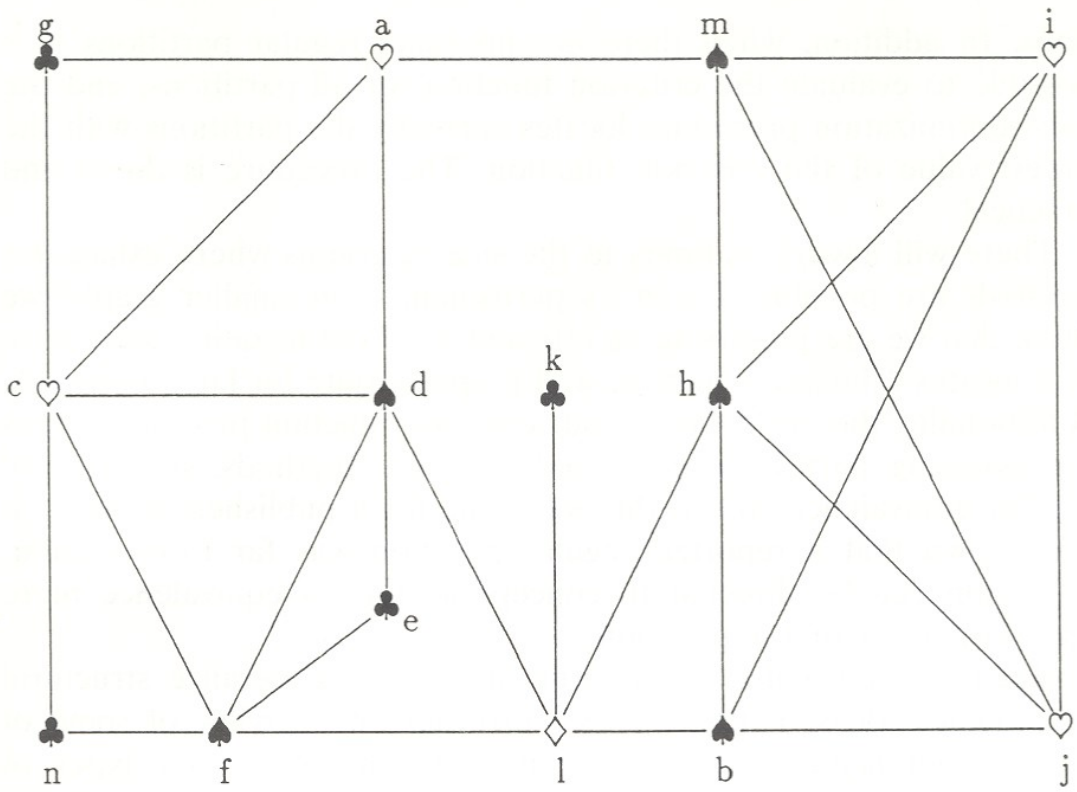


Fig. 7. Political actors – the nonregular partition ($k = 4$).

partition is *not* regular. However, that partition has an intuitively appealing interpretation in terms of the connective role of the nodes, given a prior partition in terms of structural equivalence. In the partition of Figure 7 there is the core node l, the boundary (between political alliances) spanners, $\{b, d, f, h, m\}$, the within alliance integrators $\{a, c, i, j\}$ and the within alliance minnions $\{e, g, k, n\}$. The partitions in Figures 4 to 6 lack such an appeal. To the extent that Figure 7 is an appropriate partition, it seems reasonable to look for another equivalence relational property as its foundation.

5. Conclusion

We have proposed a criterion function that expresses directly the idea of regular equivalence and have mobilized a local optimization procedure to locate regular partitions and/or near regular partitions. For graphs with 12 or less nodes, where it is possible to obtain all regular partitions, the local optimization procedure locates all regular parti-

tions. In addition, when there are no exact regular partitions it is possible to evaluate the criterion function for all partitions, and the local optimization procedure locates correctly the partitions with the lowest value of the criterion function. The procedure is direct and practical.

There will always be limits to the size of graphs where exhaustive methods are possible. Given its performance on smaller graphs we think that we are proposing an efficient local optimization procedure that locates validated partitions and is appropriate for large networks. Additionally, the evaluation of our criterion function provides a basis for assessing partitions, generated by other methods, in terms of regular equivalence. When this was done for a published network, it was shown that a reported regular partition was far from regular. Attention can be directed to conceptualizing an equivalence more appropriate for that application.

Regular equivalence is only one attempt to generalize structural equivalence ideas. Pattison (1988) provides an overview of some of these generalizing efforts. We conjecture that for other types of equivalence, a criterion function can be constructed, local optimization methods can be used to locate partitions, and that partitions can be evaluated by use of the criterion function.

Appendix – Proof of the proposition

Let us first prove that in each row of a nonnull block there is at least one 1. Suppose that the block $R(C_u, C_v)$ is not null. Then there exist $X \in C_u$ and $Z \in C_v$ such that $r_{xz} = 1$ (i.e., XRZ). If $C_u = \{X\}$ the proposition follows; otherwise take $Y \in C_u$, $Y \neq X$. From the definition of $\mathscr{C}$ it follows $X \cong Y$ and further, by $R1$, there exists $W \in E$ such that YRW and $W \cong Z$ or equivalently $r_{yw} = 1$ and $W \in C_v$. Therefore we have at least one 1 also in the row corresponding to Y in the block $R(C_u, C_v)$. By a similar argument we prove, using $R2$, the property also for columns.

To prove the converse let us first prove that the condition $R1$ holds. Let $X, Y \in G_u$. If the block $R(C_u, C_v)$ is null the condition $R1$ holds vacuously; otherwise there exist $Z, W \in C_v$ (therefore $Z \cong W$) such that $r_{xz} = 1$ and $r_{yw} = 1$ (i.e., XRZ and YRW). The condition $R1$ holds. By similar argument we prove also that the condition $R2$ holds.

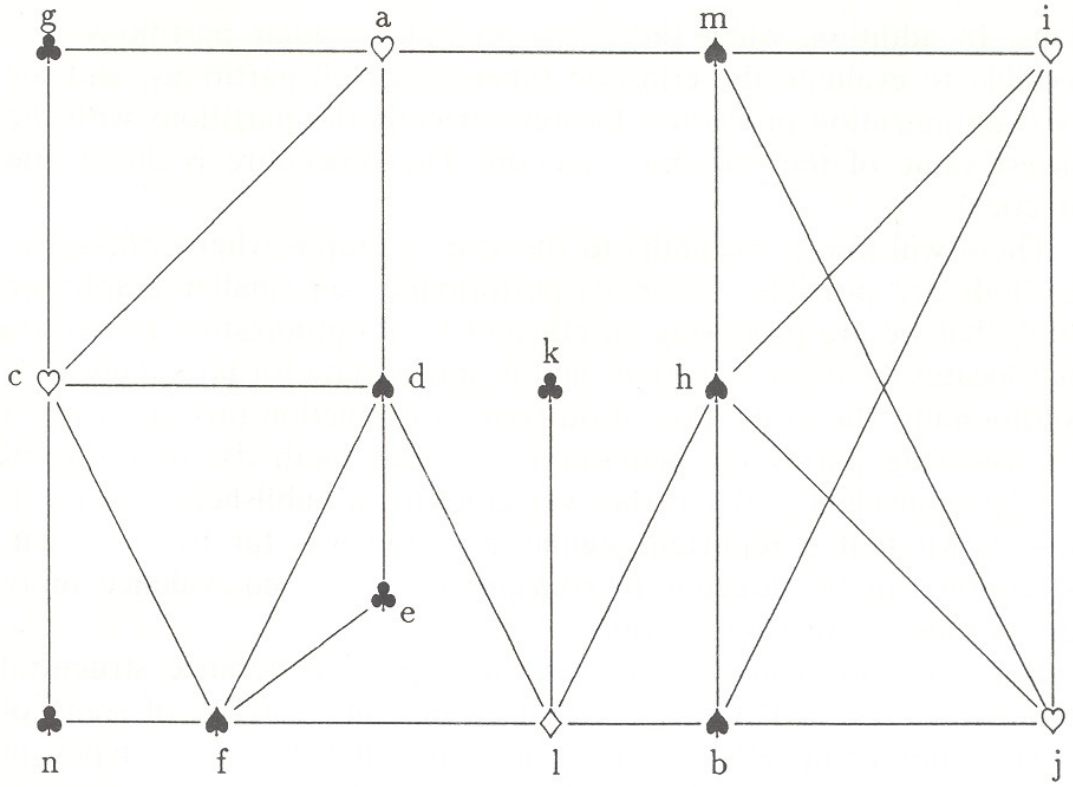


Fig. 7. Political actors – the nonregular partition ($k = 4$).

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