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COMMUNICATION

HAMILTONIAN CYCLES IN THE CARTESIAN PRODUCT OF A TREE AND A CYCLE*

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Theorem. *Let $G = T \times C_n$ be the cartesian product of an n-cycle C_n and a tree T with the maximum degree $\Delta(T) \geq 2$. Then G possesses a Hamiltonian cycle if and only if $\Delta(T) \leq n$.*

Proof. We label the vertices of the cycle C_n by the elements of the cyclic group $\mathbb{Z}_n$ in such a way that the vertex i is adjacent to $i+1$ where "plus" is taken in $\mathbb{Z}_n$. The vertices of $G = T \times C_n$ are ordered pairs of the form $(t, i) \in V(G) = V(T) \times V(C_n)$. Such a vertex will be denoted by t^i for short.

Let u be any vertex of T having maximum degree: $\deg(u) = \Delta(T)$. Furthermore, let the set S be defined as follows.

$$S = \{u^i \in V(G) \mid i \in \mathbb{Z}_n\}.$$

Clearly S has exactly n elements. By H we denote the graph obtained from G by removing the n vertices in the set S. Obviously H is disconnected and has at least $\Delta(T)$ components. If $n < \Delta(T)$ then G has no Hamiltonian cycle, (see for instance [1, Theorem 3.1]). It remains to construct a Hamiltonian cycle in the case $n \geq \Delta(T)$. Here is a construction! Take an arbitrary edge-coloring of T with n colors. For notational convenience we choose the color set to be $\mathbb{Z}_n$. The n-edge-coloring of T exists as $n \geq \Delta(T)$ and the tree T is a bipartite graph which is $\Delta(T)$-edge-colorable (see for instance [3, Theorem 2.2]).

Let $V(T) = V^- \cup V^+$ be the bipartition of T. Define a permutation $h : V(G) \to V(G)$ by the following rules:

(i) If an edge st of T with $s \in V^-$ and $t \in V^+$ is colored with $i \in \mathbb{Z}_n$, then let $h(s^i) = t^i$ and let $h(t^{i+1}) = s^{i+1}$.

(ii) If $s \in V^-$ and there is no edge st colored with $i \in \mathbb{Z}_n$, then let $h(s^i) = s^{i+1}$.

(iii) If $t \in V^+$ and there is no edge st colored with $i \in \mathbb{Z}_n$, then let $h(t^{i+1}) = t^i$.

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The reader may verify that under the assumptions above h is a cyclic permutation defining a Hamiltonian cycle in G. □

If the complete graph K_n is substituted for C_n in the statement of the theorem we obtain a similar but in a sense weaker result of Zaretsky [6]. Since $\Delta(T) < |V(T)| \leq 2|V(T)| - 2$ (for $|V(T)| \geq 2$) the theorem implies an observation by Sabidussi [5] and some results of Rosenfeld and Barnette [4].

For an arbitrary connected graph G define the *cyclic Hamiltonicity* $\mathrm{cH}(G)$ as a minimal n for which $G \times C_n$ is Hamiltonian. $\mathrm{cH}(G) = 1$ if and only if G is Hamiltonian. For a tree T our theorem shows that $\mathrm{cH}(T) = \Delta(T)$. Let us define for a connected graph G the number $\nabla(G)$ as

$$\nabla(G) = \min\{\Delta(T) \mid T \text{ is a spanning tree for } G\}.$$

Clearly $\nabla(G) \leq \Delta(G)$. Our theorem implies

$$(*) \qquad \mathrm{cH}(G) \leq \nabla(G).$$

There exist graphs with arbitrary large cyclic Hamiltonicity for which strict inequality holds in $(*)$. For instance for the complete bipartite graph $K_{2,2r}$ we have $\mathrm{cH}(K_{2,2r}) = r$, yet $\nabla(K_{2,2r}) = r + 1$. It is possible to prove that $\mathrm{cH}(G) \leq \nabla(G) \leq \mathrm{cH}(G) + 1$. From this inequality it follows that a Hamiltonian cycle exists in $G \times C_n$ for each $n \geq \mathrm{cH}(G)$. The proofs will appear elsewhere.

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