From Fractal Image Compression to ODEs: Solving Inverse Problems Using Contraction Maps

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Fractal-based image coding methods seek to approximate a target image $v(x)$ with the fixed point attractor function $\bar{u}(x)$ of a contractive *fractal transform* operator T that acts upon a suitable metric space $(\mathcal{F}, d_{\mathcal{F}})$ of image functions. The action of the fractal transform T on a function u involves (1) creating a number of contracted copies of u, (2) modifying these copies and (3) recombining them to form a new function Tu. The problem is to find the "best" such operator T. The parameters that define T - the so-called *fractal code* - are then used to represent the target image v. The approximation $\bar{u}$ is generated by the iteration sequence $u_{n+1} = Tu_n$, where $u_0 \in \mathcal{F}$ is a suitable "seed" (for example, a blank screen). Under suitable conditions, the computer storage of the fractal code requires much less memory than that of the original image. The result, therefore, is *lossy image compression* since full accuracy is generally not achieved.

More generally, fractal-based coding involves the approximation of elements of an appropriate metric space by fixed points of contractive operators on that space. As expected, Banach's Fixed Point Theorem is the basis of this method along with an important corollary that has been called the "Collage Theorem."

It has been our goal to look for other, possibly "nonfractal," inverse problems in which these methods can be employed. One such example is a simple class of inverse problems involving ordinary differential equations: Given a function $x(t)$ (which may be an interpolation of a set of experimental data points (x_i, t_i)), find an ODE $\dot{x} = f(x, t)$ that best admits $x(t)$ as an exact or approximate solution, where f is restricted to a class of functional forms, e.g. polynomial. (For example, a population biologist may wish to find the best Lotka-Volterra system that admits a set of experimental results as solution.) In this case, we use the Picard contraction mapping and, as in fractal-based methods, find the Picard map T (in turn defined by f) that maps the target $x(t)$ as close as possible to itself. It turns out that such inverse problems have received much interest in areas such as population dynamics, (radioactive) tracer dynamics and chemical kinetics. Indeed, the various numerical schemes that have been reported in the literature are essentially based on the above "collage method." As such, we have now been able to mathematically justify these various methods.

This talk, aimed at a general audience, will outline the major points behind the "approximation by fixed points problem," using fractal coding as an illustrative example. We then move on to the above-mentioned inverse problem for ODEs and show how the Picard integral operator is used to construct optimal vector fields. Some simple examples in one and two dimensions are then presented.