Fitting Pre-specified Blockmodels

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There are many inductive approaches for establishing blockmodels for a set of social relations defined over a set of social actors. Some form of equivalence is specified and clusterings are sought that are consistent with a specified equivalence. Even more generalized clusterings can be located. In all cases, the analyses respond to empirical information.

Another view of blockmodeling is deductive in the sense of starting with a blockmodel that is specified in terms of substance prior to an analysis. In this paper we present methods where a set of observed relations are fitted to a pre-specified blockmodel. These methods are illustrated with a set of empirical examples.

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Blockmodeling

The goal of blockmodeling is to reduce a large, potentially incoherent network to a smaller comprehensible structure that can be interpreted more readily. Blockmodeling, as an empirical procedure, is based on the idea that units in a network can be grouped according to the extent to which they are equivalent, according to some *meaningful* definition of equivalence.

Example: Student Government

The analyzed network consists of communication interactions among twelve members and advisors of the Student Government at the University in Ljubljana (Tina Hlebec,1993). The results of the measurement are not real interactions among actors but cognition about communication interactions.

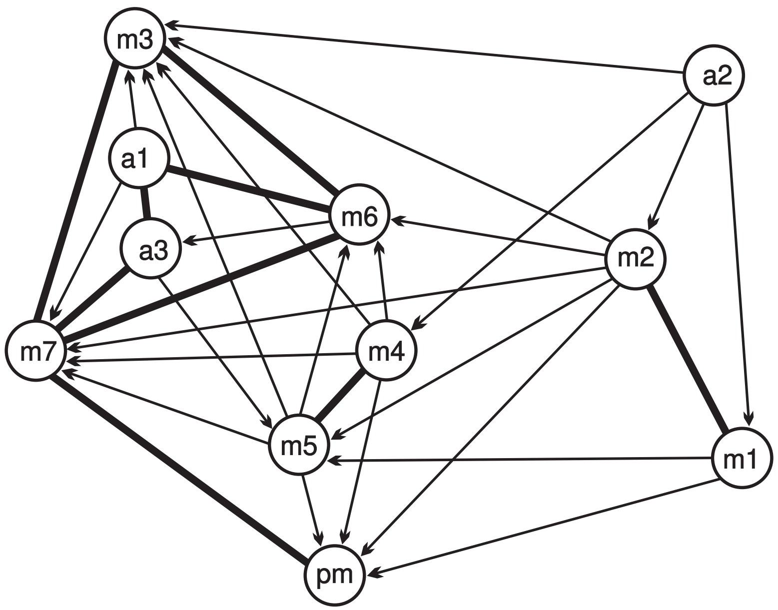
Data were collected with face to face interviews. Interviews were conducted in May 1992.

Communication flow among actors was identified by the following question:

> Of the members and advisors of the Student Government, whom do you (most often) talk with?

The content of the communication flow was limited to the matters of the Student Government. The time frame was also defined: the question was referred to the six months period. One respondent refused to cooperate in the experiment. As he was not considered in the analysis, the network consists of eleven actors.

Student Government Network



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Student Government Matrix

		m 1	p 2	m 3	m 4	m 5	m 6	m 7	m 8	a 9	a 10	a 11
minister 1	1	0	1	1	0	0	1	0	0	0	0	0
p.minister	2	0	0	0	0	0	0	0	1	0	0	0
minister 2	3	1	1	0	1	0	1	1	1	0	0	0
minister 3	4	0	0	0	0	0	0	1	1	0	0	0
minister 4	5	0	1	0	1	0	1	1	1	0	0	0
minister 5	6	0	1	0	1	1	0	1	1	0	0	0
minister 6	7	0	0	0	1	0	0	0	1	1	0	1
minister 7	8	0	1	0	1	0	0	1	0	0	0	1
adviser 1	9	0	0	0	1	0	0	1	1	0	0	1
adviser 2	10	1	0	1	1	1	0	0	0	0	0	0
adviser 3	11	0	0	0	0	0	1	0	1	1	0	0

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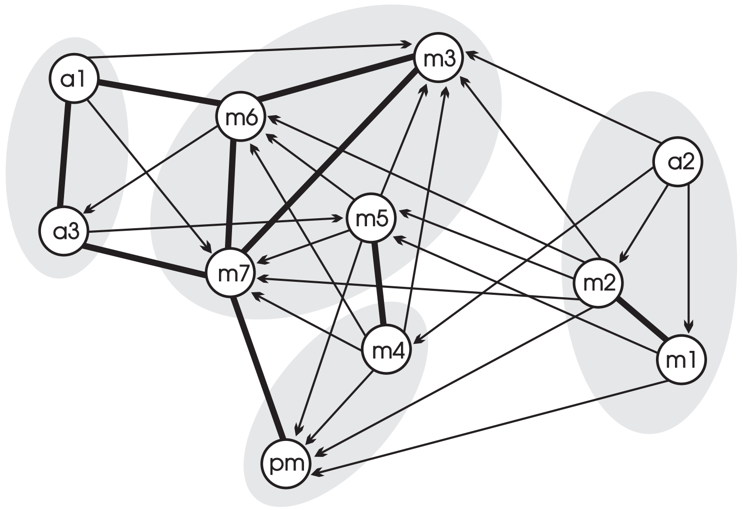
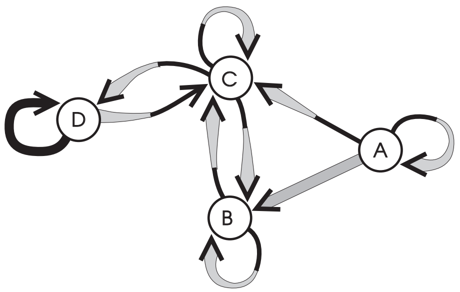
Blockmodel





Basic Notions

Network

Let $E = \{X_1, X_2, \ldots, X_n\}$ be a finite set of *units or units.* The units are related by binary *relations*

$$R_t \subseteq E \times E, \quad t = 1, \ldots, r$$

which determine a *network*

$$\mathcal{N} = (E, R_1, R_2, \ldots, R_r)$$

In the following we restrict our discussion to a single relation R described by a corresponding binary matrix $\mathbf{R} = [r_{ij}]_{n \times n}$ where

$$r_{ij} = \begin{cases} 1 & X_i R X_j \\ 0 & \text{otherwise} \end{cases}$$

In some applications r_{ij} can be a nonnegative real number expressing the strength of the relation R between units X_i and X_j.

Cluster, clustering

One of the main procedural goals of blockmodeling is to identify, in a given network, *clusters* (classes) of units that share structural characteristics defined in terms of R. The units within a cluster have the same or similar connection patterns to other units. They form a *clustering*

$$\mathcal{C} = \{C_1, C_2, \ldots, C_k\}$$

which is a partition of the set E:

$$\bigcup_i C_i = E$$

$$i \neq j \Rightarrow C_i \cap C_j = \emptyset$$

Each partition determines an equivalence relation (and vice versa). Let us denote by $\sim$ the relation determined by partition $\mathcal{C}$.

Block

A clustering $\mathcal{C}$ partitions also the relation R into *blocks*

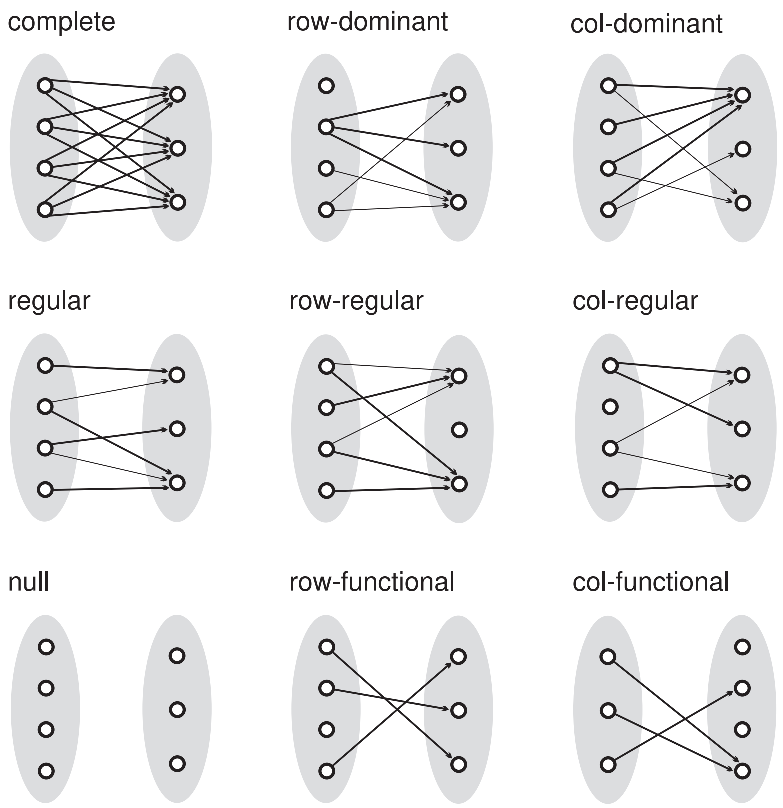
$$R(C_i, C_j) = R \cap C_i \times C_j$$

Each such block consists of units belonging to clusters C_i and C_j and all arcs leading from cluster C_i to cluster C_j. If $i = j$, a block $R(C_i, C_i)$ is called a *diagonal* block.

Blockmodel

A *blockmodel* consists of structures obtained by identifying all units from the same cluster of the clustering $\mathcal{C}$. For an exact definition of a blockmodel we have to be precise also about which blocks produce an arc in the *reduced graph* and which do not. The reduced graph can be presented by relational matrix, called also *image matrix*.

Block Types



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Block Types and Matrices

$$
\left.\begin{array}{cccc|cccc}
1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0
\end{array}\right.
$$

complete	regular
null	complete

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Blockmodeling - Formalization

A *blockmodel* is an ordered sextuple $\mathcal{M} = (U, K, \mathcal{T}, Q, \pi, \alpha)$ where:

- U is a set of *types* of units (images or representatives of classes);
- $K \subseteq U \times U$ is a set of *connections*;
- $\mathcal{T}$ is a set of predicates used to describe the types of connections between different classes (clusters, groups, types of units) in a network. We assume that $\text{nul} \in \mathcal{T}$. A mapping $\pi : K \to \mathcal{T} \setminus \{\text{nul}\}$ assigns predicates to connections;
- Q is a set of *averaging rules*. A mapping $\alpha : K \to Q$ determines rules for computing values of connections.

For $\mu : V \to U$ we define for $t \in U$

$$C(t) = \mu^{-1}(t) = \{x \in V : \mu(x) = t\}.$$

Therefore $\mathcal{C}(\mu) = \{C(t) : t \in U\}$ is a partition (clustering) of the set of units V.

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A (surjective) mapping $\mu : V \to U$ determines a blockmodel $\mathcal{M}$ of network $\mathcal{N}$ iff it satisfies the conditions:

$$\forall (t, w) \in K : \pi(t, w)(C(t), C(w))$$

and

$$\forall (t, w) \in U \times U \setminus K : \text{nul}(C(t), C(w)).$$

Note, $\mathcal{T} = \{\text{nul}, \text{com}\} \Rightarrow$ a structural blockmodel (Lorrain and White 1971);
and, $\mathcal{T} = \{\text{nul}, \text{reg}\} \Rightarrow$ a regular blockmodel (White and Reitz 1983).

Equivalences

Let $\approx$ be an equivalence relation over V and

$$[x] = \{y \in V : x \approx y\}.$$

We say that $\approx$ is *compatible* with $\mathcal{T}$ over a network $\mathcal{N}$ iff

$$\forall x, y \in V \exists T \in \mathcal{T} : T([x], [y]).$$

It is easy to verify that the notion of compatibility for $\mathcal{T} = \{\text{nul}, \text{reg}\}$ reduces to the usual definition of regular equivalence (Borgatti and Everett 1989).

For a compatible equivalence $\approx$ the mapping $\mu : x \mapsto [x]$ determines a blockmodel.

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Optimization

A criterion function

One of the possible ways of constructing a criterion function that directly reflects the considered equivalence is to measure the fit of a clustering to an ideal one with perfect relations within each cluster and between clusters according to the considered equivalence.

Given a clustering $\mathcal{C} = \{C_1, C_2, \ldots, C_k\}$, let $\mathcal{B}(C_u, C_v)$ denote the set of all ideal blocks corresponding to block $R(C_u, C_v)$. Then the global error of clustering $\mathcal{C}$ can be expressed as

$$P(\mathcal{C}) = \sum_{C_u, C_v \in \mathcal{C}} \min_{B \in \mathcal{B}(C_u, C_v)} d(R(C_u, C_v), B)$$

where the term $d(R(C_u, C_v), B)$ measures the difference (error) between the block $R(C_u, C_v)$ and the ideal block B. The function d has to be compatible with the selected type of equivalence.

Given a set of types of connection $\mathcal{T}$ and a block $L(X, Y)$, we can determine the strongest (according to the ordering of the set $\mathcal{T}$) type T which is satisfied by $L(X, Y)$. In this case we set

$$\pi(\mu(X), \mu(Y)) = T$$

But what is to be done, if no type from $\mathcal{T}$ is satisfied?

We can introduce the set of *ideal blocks* for a given type $T \in \mathcal{T}$

$$\mathcal{B}(X, Y; T) = \{B \subseteq K(X, Y) : T(B)\}$$

and define the *deviation* $\delta(X, Y; T)$ of a block $L(X, Y)$ from the nearest ideal block. We can efficiently test whether the block $L(X, Y)$ is of the type T.

null	all 0 (except may be diagonal)
complete	all 1 (except may be diagonal)
row-dominant	$\exists$ all 1 row (except may be diagonal)
col-dominant	$\exists$ all 1 column (except may be diagonal)
regular	1-covered rows and 1-covered columns

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On the basis of these characterizations we can also construct the corresponding measures of deviation from the ideal realization.

For the proposed types all deviations are *sensitive*

$$\delta(X,Y;T) = 0 \Leftrightarrow T(L(X,Y)).$$

Therefore a block $L(X,Y)$ is of a type T exactly when the corresponding deviation $\delta(X,Y;T)$ is 0.

In the deviation δ we can also incorporate values of lines ν.

Based on deviation $\delta(X,Y;T)$ we introduce the *block-error* $\varepsilon(X,Y;T)$ of $L(X,Y)$ for type T. Two examples of block-errors are

$$\varepsilon_1(X,Y;T) = w(T)\delta(X,Y;T)$$

and

$$\varepsilon_2(X,Y;T) = \frac{w(T)}{n_r n_c}(1+\delta(X,Y;T)),$$

where $w(T) > 0$ is a weight of type T.

We extend the block-error to the set of feasible types $\mathcal{T}$ by defining

$$\varepsilon(X,Y;\mathcal{T}) = \min_{T\in\mathcal{T}} \varepsilon(X,Y;T)$$

and

$$\pi(\mu(X),\mu(Y)) = \mathrm{argmin}_{T\in\mathcal{T}}\varepsilon(X,Y;T)$$

To make π well-defined, we order (priorities) the set $\mathcal{T}$ and select the first type from $\mathcal{T}$ which minimizes ε.

We combine block-errors into a *total error* – blockmodeling *criterion function*

$$P(\mu;\mathcal{T}) = \sum_{(t,w)\in U\times U} \varepsilon(C(t),C(w);\mathcal{T}).$$

The criterion functions based on block-errors ε_1 and ε_2 are denoted P_1 and P_2 respectively.

For criterion function $P_1(\mu)$ we have

$$P_1(\mu) = 0 \quad \Leftrightarrow \quad \mu \text{ is an exact blockmodeling}$$

Also for P_2, we obtain an exact blockmodeling μ iff the deviations of all blocks are 0.

The obtained optimization problem can be solved by local optimization

Once a partitioning μ and types of connection π are determined, we can also compute the values of connections by using *averaging rules*.

Local Optimization

For solving the blockmodeling problem we use a local optimization procedure (relocation algorithm):

Determine the initial clustering $\mathcal{C}$;
repeat:
    if in the neighborhood of the current clustering $\mathcal{C}$
    there exists a clustering $\mathcal{C}'$ such that $P(\mathcal{C}') < P(\mathcal{C})$
    then move to clustering $\mathcal{C}'$.

The neighborhood in this local optimization procedure is determined by the following two transformations:

- *moving* a unit X_k from cluster C_p to cluster C_q (*transition*);
- *interchanging* units X_u and X_v from different clusters C_p and C_q (*transposition*).

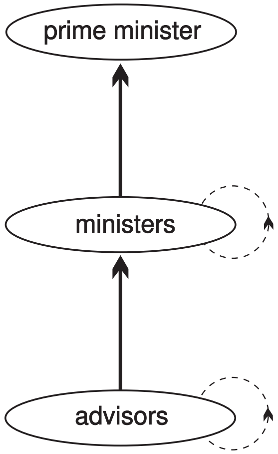
Benefits from Optimization Approach

- *ordinary / inductive blockmodeling*: Given a network $\mathcal{N}$ and set of types of connection $\mathcal{T}$, determine $\mathcal{M}$, i.e., μ, π and α;
- *evaluation of the quality of a model, comparing different models, analyzing the evolution of a network* (Sampson data, Doreian and Mrvar 1994): Given a network $\mathcal{N}$, a model $\mathcal{M}$, and blockmodeling μ, compute the corresponding criterion function;
- *model fitting / deductive blockmodeling*: Given a network $\mathcal{N}$, set of types $\mathcal{T}$, and a model $\mathcal{M}$, determine μ which minimizes the criterion function.
- we can fit the network to a partial model and analyze the residual afterward;
- we can also introduce different constraints on the model, for example: units x and y are of the same type; or, types of units x and y are not connected; ...

Fitting - Example

Hypothetical Blockmodel

The hypothetical structure of the Student Government network is a hierarchy , where each advisor communicates at least to one minister and the ministers to the prime minister.



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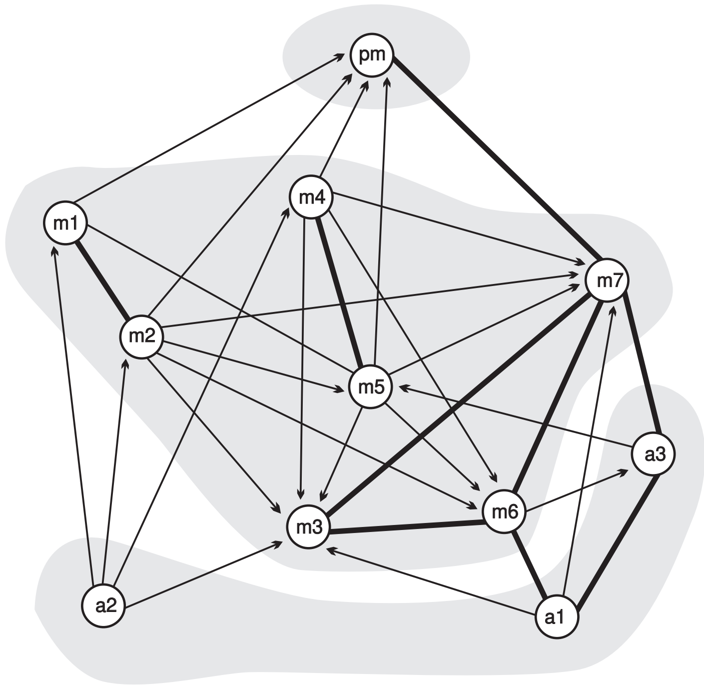
Hypothetical Structure



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Inductive Approach

- 3 clusters;
- regular equivalence

Result: many regular solutions with 7 errors.

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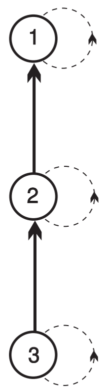
Deductive Approach

- 3 clusters;
- regular blocks
- pre-specified image matrix

	1	2	3
1	{ 0, reg }	{ 0 }	{ 0 }
2	{ reg }	{ 0, reg }	{ 0 }
3	{ 0 }	{ reg }	{ 0, reg }



Result: at least 5 different solutions with 7 errors.

`{ pm, m7, a3 },`
`{ m1, m2, m3, m4, m5, m6, a1 },`
`{ a2 }`

	1	2	3
1	reg	0	0
2	reg	reg	0
3	0	reg	0

	1	2	3
1	0	4	0
2	0	0	0
3	0	3	0

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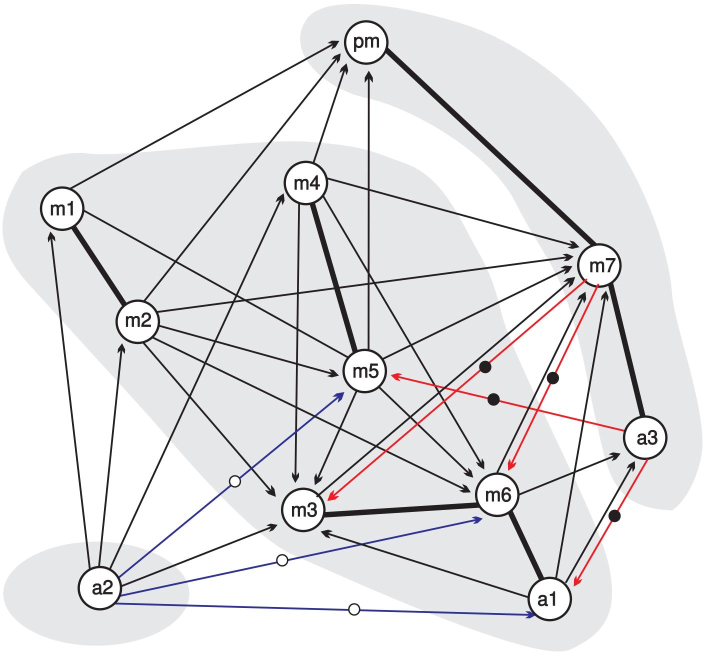
First Deductive Solution



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Second Deductive Solution

- same pre-specified model;
- constraints on units in clusters: all advisers are in cluster 3

Result: single solution with 8 errors.

```
{ pm },
{ m1, m2, m3, m4, m5, m6, m7 },
{ a1, a2, a3 }
```

	1	2	3
1	0	0	0
2	reg	reg	0
3	0	reg	0

	1	2	3
1	0	1	0
2	2	0	3
3	0	0	2

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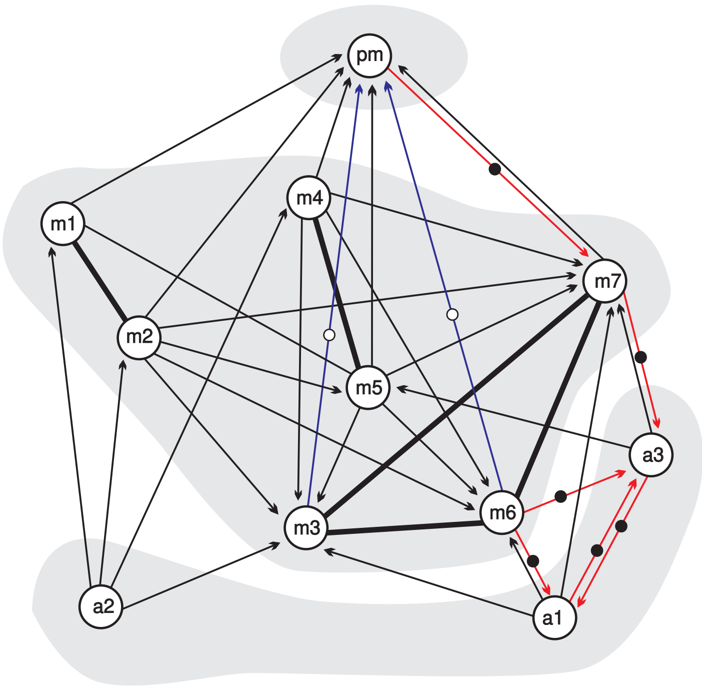

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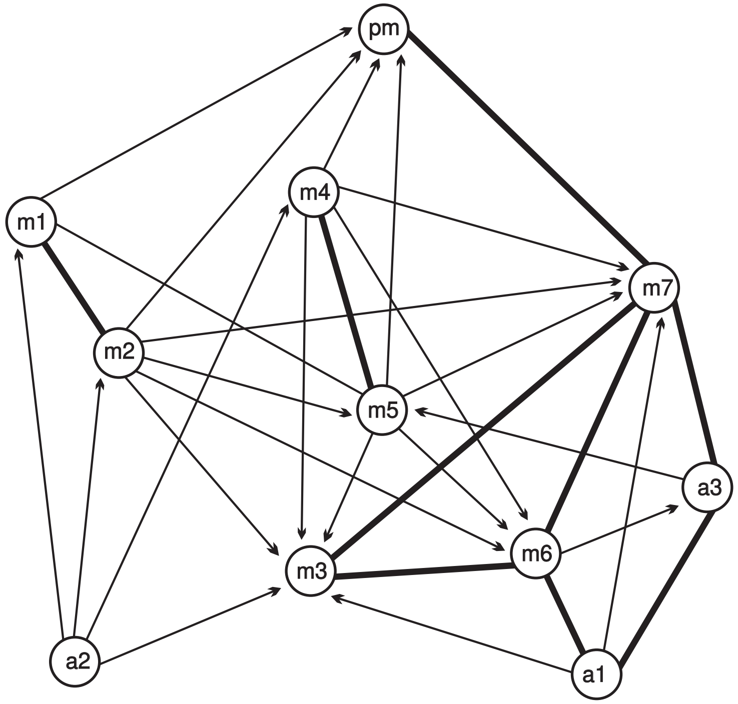

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