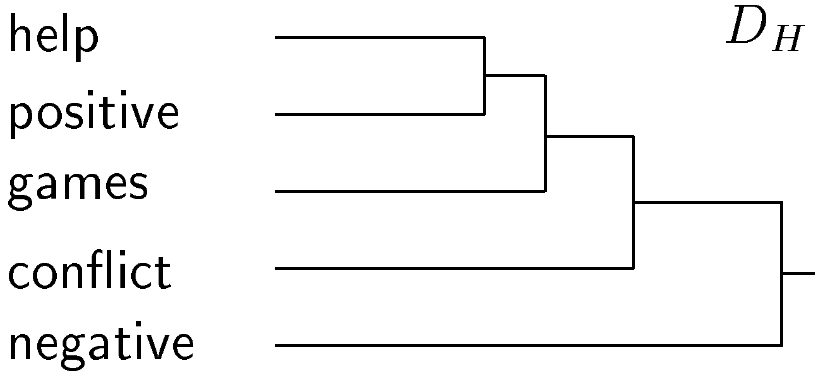
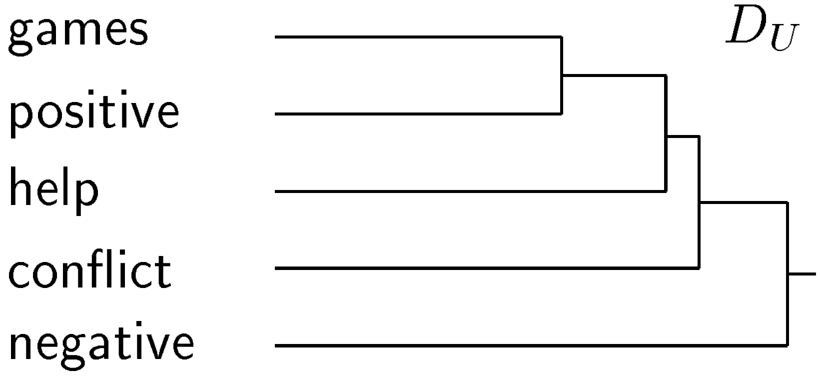
CLUSE – ward; BWR – relational dissimilarities

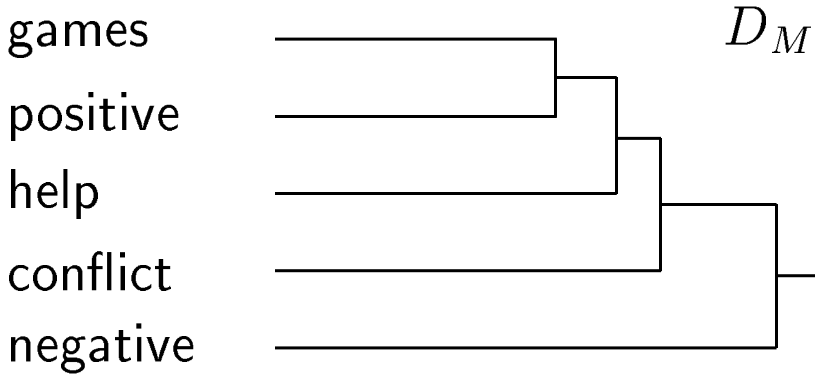
D_H $[0.00, 0.42]$



D_U $[0.00, 1.10]$



D_M $[0.00, 1.10]$



4. Example: Semigroup Homomorphisms

Let us, for an example take two relations P and N (Bonacich P., McConaghy M.J.: *The Algebra of Blockmodeling*, p. 516-520). These two relations generate the multiplication semigroup consisting of relations P and N and all their products: P,N,NP,PN,N^2.

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad N = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

	P	N	PN	NP	N^2
P	P	PN	PN	N^2	N^2
N	NP	N^2	N^2	N^2	N^2
PN	N^2	N^2	N^2	N^2	N^2
NP	NP	N^2	N^2	N^2	N^2
N^2	N^2	N^2	N^2	N^2	N^2

We want to simplify this semigroup by mapping it onto less complex semigroup using a homomorphism.

Some compound relations are defined to be equal. Any equation among semigroup elements implies a new set of congruence classes. The error inside congruence classes is measured by the maximum distance between any pair of relations in the class.

Congruence classes	Equation	D_H	D_U	D_M
1. $/P,N,NP,PN,N^2/$	$P = N$	8	0.89	0.83
2. N / P,NP,PN,N^2	$P = NP$ or $P = PN$ or $P = N^2$	7	0.78	0.71
3. P / N,NP,PN,N^2	$N = N^2$	3	0.33	0.33
4. P / N / NP,PN,N^2	$PN = NP$	2	0.25	0.22
5. P / N,PN / NP,N^2	$N = PN$	2	0.22	0.22
6. P / N,NP / PN,N^2	$N = NP$	2	0.22	0.22
7. P / N / NP / PN,N^2	$PN = N^2$	2	0.22	0.22
8. P / N / PN / NP,N^2	$NP = N^2$	2	0.22	0.22
9. P / N / PN / NP / N^2	–	0	0	0

The most preferred homomorphism is selected using error (distance) and the number of congruence classes. In this way D_H and D_M distances suggest that the best homomorphisms are 4, 5 and 6, while using D_U distance only 5 and 6 are equally preferred.

	D_H					D_U					D_M				
	N^2	PN	NP	N	P	N^2	PN	NP	N	P	N^2	PN	NP	N	P
N^2	0					0					0				
PN	2	0				0.22	0				0.22	0			
NP	2	2	0			0.22	0.25	0			0.22	0.14	0		
N	3	1	1	0		0.33	0.14	0.14	0		0.33	0.14	0.14	0	
P	5	7	7	8	0	0.56	0.78	0.78	0.89	0	0.56	0.71	0.71	0.83	0

5. Example: Reachability Problems

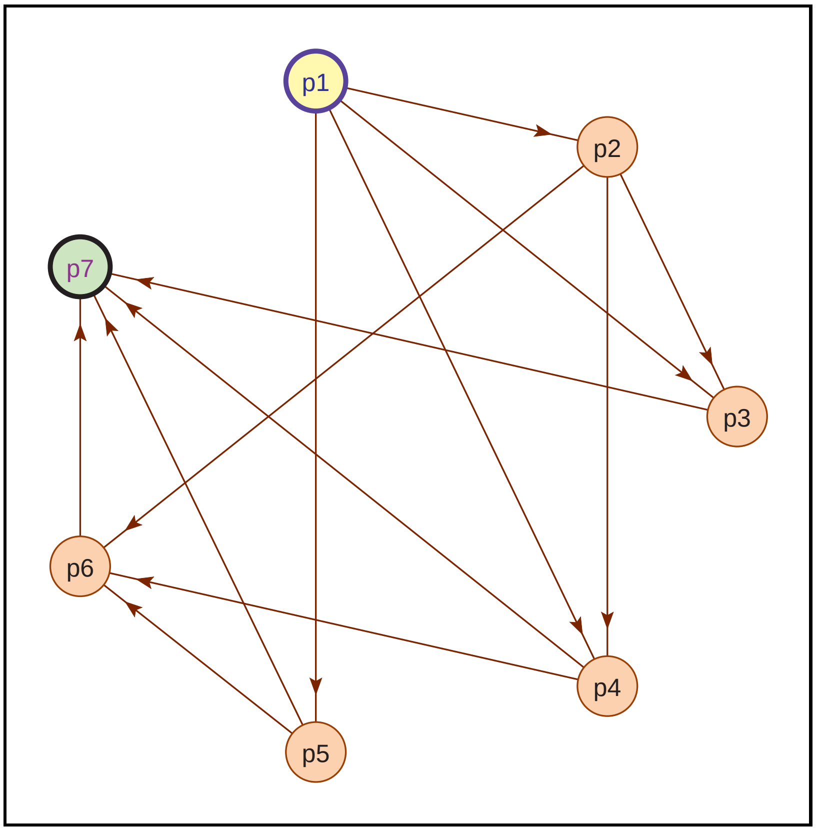
Using relational algebra changes in state of certain systems can be investigated. The problem of two missionaries and two cannibals is a well known puzzle (Andrasfai B.: Graph Theory: Flows, Matrices).

> Two missionaries and two cannibals reach the left bank of a river simultaneously and all of them want to cross the river with the aid of the single available boat which carries no more than two persons. On no bank of the river may the cannibals outnumber the missionaries otherwise the former jeopardize the latter.

The problem can be described using two sets:

- Set of all feasible states – states where the number of missionaries on both banks of the river is higher or equal to the number of cannibals.
- Set of possible transitions between states. These transitions tell us which states can be obtained from a given state by crossing the river only once.

Each feasible state can be represented by a pair of numbers: number of missionaries and number of cannibals on the left bank of the river: $p_1 = (2, 2)$, $p_2 = (2, 1)$, $p_3 = (2, 0)$, $p_4 = (1, 1)$, $p_5 = (0, 2)$, $p_6 = (0, 1)$, $p_7 = (0, 0)$.



5.1 Solution of the problem using `RelCalc`

States p_1 to p_7 can be represented using following clusterings:

$$p_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad p_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad p_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \ldots \quad p_7 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

and the graph by the matrix

$$B(R) = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

To solve the puzzle we have to answer the questions:

1. Does the problem have any solution?
2. What is the shortest solution?

The problem has the solution iff there exists a direction altering trail from p_1 to p_7. This is equivalent to $(p_1, p_7) \in (RR^{-1})^* R$.

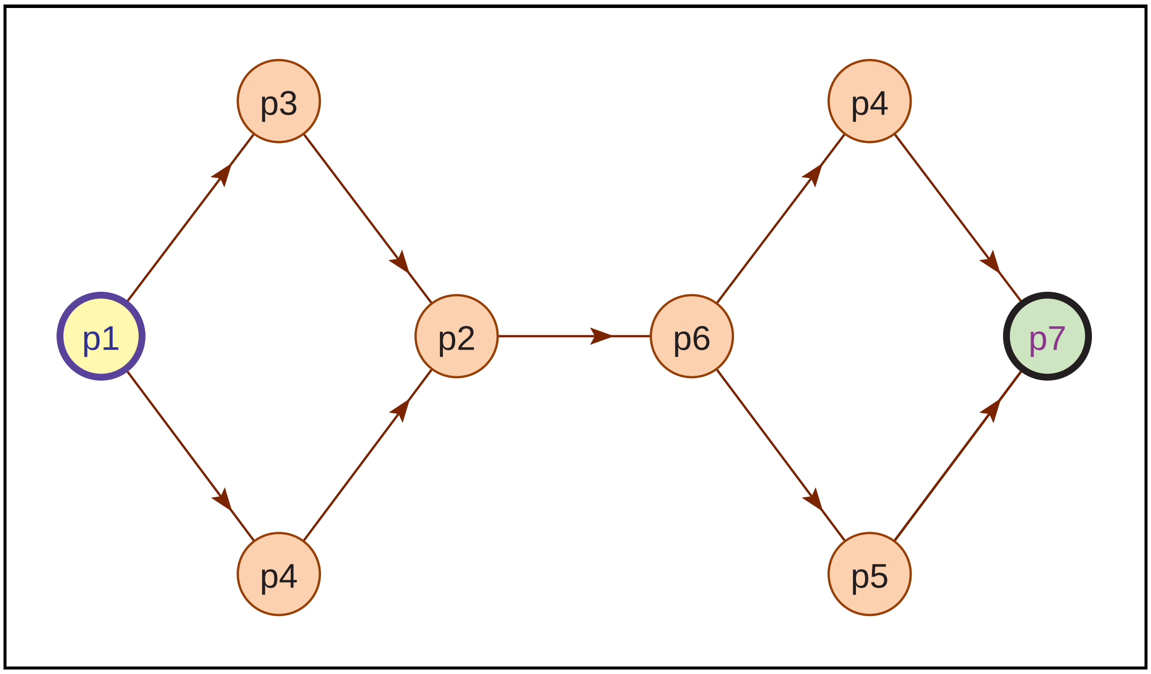
$$B((RR^{-1})^* R) = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The problem has a solution. We can answer the second question by computing series of clusterings $(RR^{-1})^k R(p_1)$ and look for the first clustering in which the final state p_7 appears. The computation gives $k = 2$ – the lowest number of crossings is 5.

There are different solutions of the problem. Let $P_i = \underbrace{RR^{-1}R\cdots}_{i \text{ terms}}$ and $Q_i = \underbrace{R^{-1}RR^{-1}\cdots}_{2k+1-i \text{ terms}}$, $i = 1, \ldots, 2k$, then the set S_i of states, i steps from start, belonging to the solutions is determined by

$$S_i = P_i(p_1) \cap Q_i(p_7)$$

There are 4 solutions:



$p_1 \rightarrow p_3 \rightarrow p_2 \rightarrow p_6 \rightarrow p_4 \rightarrow p_7$

$p_1 \rightarrow p_3 \rightarrow p_2 \rightarrow p_6 \rightarrow p_5 \rightarrow p_7$

$p_1 \rightarrow p_4 \rightarrow p_2 \rightarrow p_6 \rightarrow p_4 \rightarrow p_7$

$p_1 \rightarrow p_4 \rightarrow p_2 \rightarrow p_6 \rightarrow p_5 \rightarrow p_7$