Short Cycle Connectivity

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Abstract

Short cycle connectivity is a generalization of ordinary connectivity. Instead of by a path (sequence of edges), two vertices have to be connected by a sequence of short cycles, in which two consecutive cycles have at least one common vertex. If all consecutive cycles in the sequence share at least one edge, we talk about edge short cycle connectivity. Short cycle connectivity can be extended to directed graphs (cyclic and transitive connectivity).

It is shown that the short cycle connectivity is an equivalence relation on the set of vertices, while the edge/arc short cycle connectivity components determine an equivalence relation on the set of edges/arcs. Some additional properties of these relations are also presented.

The related notion of short cycle networks provides us with a tool for identification of dense parts of graphs with applications in the design of algorithms and social network analysis (hierarchies, Granovetter's strong and weak ties). For further generalization we can also consider connectivity by small cliques or other families of graphs.

1 Introduction

The idea of connectivity by short cycles emerges in different contexts. In hierarchical decompositions of networks the long cycles can be violations of the assumed hierarchical structure. The symmetric connectivity from paper

[5] is essentially the connectivity by 2-cycles. Edges/arcs belonging to short cyclic components can be considered as 'strong' ties [4]; 'weak' ties linking an individual to other 'groups' (components) are important for her/his success in accessing different resources (job, information, etc.). In [1] we were looking at subgraphs formed by complete triads – triangles. Triangular connectivity also appears to be important in different applications [10, 13, 6, 14].

The next stimulus was a reference in Scott [12] to the early work of M. Everett on this subject [7, 8, 9]. It seems that his ideas can be elaborated to provide a powerful and efficient tool for analysis of large networks.

In the paper we provide a formal setting for these notions and present some of their basic properties. We first consider the short cycle connectivity in undirected graphs and afterward extend our discussion to directed graphs.

2 k-gonal connectivity in undirected graphs

Let $\mathbf{K}$ denote the *connectivity* relation and $\mathbf{B}$ denote the *biconnectivity* relation in a given simple undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Let $n = |\mathcal{V}|$ denote the number of vertices and let $m = |\mathcal{E}|$ denote the number of edges.

Vertex $u \in \mathcal{V}$ is in relation $\mathbf{K}$ with vertex $v \in \mathcal{V}$, $u\mathbf{K}v$, if and only if $u = v$ or there exists a path in $\mathcal{G}$ from u to v.

Vertex $u \in \mathcal{V}$ is in relation $\mathbf{B}$ with vertex $v \in \mathcal{V}$, $u\mathbf{B}v$, if and only if $u = v$ or there exists a cycle in $\mathcal{G}$ containing u and v.

We call a *k-gone* a subgraph isomorphic to a k-cycle C_k and a *(k)-gone* a subgraph isomorphic to C_s for some s, $3 \leq s \leq k$. A subgraph $\mathcal{H}$ of $\mathcal{G}$ is *k-gonal*, if each of its vertices and each of its edges belong to at least one (k)-gone in $\mathcal{H}$.

Definition 1 *A sequence $(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_s)$ of (k)-gons of $\mathcal{G}$ (*vertex*) k-*gonally connects *a vertex $u \in \mathcal{V}$ with a vertex $v \in \mathcal{V}$, if and only if*

1. *$u \in \mathcal{V}(\mathcal{C}_1)$,*
2. *$v \in \mathcal{V}(\mathcal{C}_s)$, and*
3. *$\mathcal{V}(\mathcal{C}_{i-1}) \cap \mathcal{V}(\mathcal{C}_i) \neq \emptyset \quad for \quad i = 2, \ldots, s$.*

*Such a sequence is called a (*vertex*) k-*gonal chain*, see Figure 1. Vertex $u \in \mathcal{V}$ is (*vertex*) k-*gonally connected* with vertex $v \in \mathcal{V}$, $u\mathbf{K}_k v$, if and only if $u = v$ or there exists a (vertex) k-gonal chain that (vertex) k-gonally connects vertex u with vertex v.*

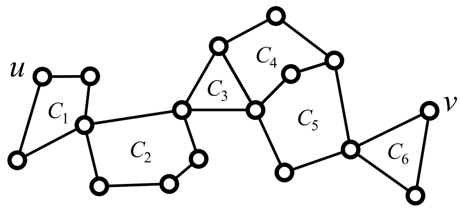

Figure 1: *5-gonal chain from u to v*

Theorem 1 *The relation $\mathbf{K}_k$ is an equivalence relation on the set of vertices $\mathcal{V}$.*

PROOF: Reflexivity follows directly from the definition of the relation $\mathbf{K}_k$.

Since the reverse of a k-gonal chain from u to v is a k-gonal chain from v to u, the relation $\mathbf{K}_k$ is symmetric.

Transitivity. Let u, v and z be such vertices, that $u\mathbf{K}_k v$ and $v\mathbf{K}_k z$. If these vertices are not pairwise different, the transitivity condition is trivially true. Assume now that they are pairwise different. Because of $u\mathbf{K}_k v$ and $v\mathbf{K}_k z$ there exist (vertex) k-gonal chains from u to v and from v to z. Their concatenation is a (vertex) k-gonal chain from u to z. Therefore also $u\mathbf{K}_k z$. □

Subgraphs induced by $\mathbf{K}_k$-equivalence classes are called *(vertex) k-gonal connectivity components*. A k-gonal connectivity component is *trivial* if and only if it consists of a single vertex.

Theorem 2 *The sets of vertices of maximal connected k-gonal subgraphs are exactly nontrivial (vertex) k-gonal connectivity classes.*

PROOF: Let u and v be any vertices belonging to a connected k-gonal subgraph. If $u = v$, it is obvious that $u\mathbf{K}_k v$. Otherwise there exists a path $\pi = u, e_1, z_1, e_2, z_2, e_3, z_3, \ldots, e_s, v$ from u to v. Because the subgraph is k-gonal, each edge e_i on this path belongs to at least one (k)-gone $\mathcal{C}_i$ in this subgraph. For the obtained k-gonal chain $(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_s)$ it holds:

- $e_i \in \mathcal{E}(\mathcal{C}_i)$, $i = 1, \ldots, s$
- $u \in \mathcal{V}(\mathcal{C}_1)$, $v \in \mathcal{V}(\mathcal{C}_s)$
- $z_{i-1} \in \mathcal{V}(\mathcal{C}_{i-1}) \cap \mathcal{V}(\mathcal{C}_i)$, $i = 2, \ldots, s$

Therefore $u\mathbf{K}_k v$. So all vertices of any (also maximal) connected k-gonal subgraph belong to the same component of the relation $\mathbf{K}_k$.

Now, let u and v be two different vertices of a nontrivial $\mathbf{K}_k$-component $\mathcal{C} \subseteq \mathcal{V}$. Because u is in relation $\mathbf{K}_k$ with v, there exists a k-gonal chain from u to v. It is obvious that all vertices of a k-gonal chain belong to the same maximal connected k-gonal subgraph, so also u and v. But u and v were any two different vertices of $\mathcal{C}$, so all vertices of a nontrivial k-gonal connectivity component belong to the same maximal connected k-gonal subgraph. □

Note that nontrivial (vertex) k-gonal connectivity components are not necessary k-gonal subgraphs and therefore they are not maximal connected k-gonal subgraphs. We can see this from the example in Figure 2, where all vertices are in the same triangular connectivity component, but the graph is not triangular because of the edge e, which does not belong to a triangle.

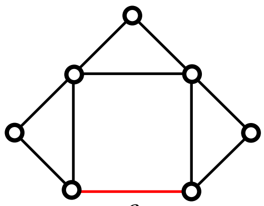

Figure 2: *This graph is not triangular*

Definition 2 *In the* k-gonal network $\mathcal{N}_k(\mathcal{G}) = (\mathcal{V}, \mathcal{E}, w_k)$ *on graph* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ *the weight* $w_k(e)$ *of an edge* $e \in \mathcal{E}$ *is equal to the number of different* (k)*-gons in* $\mathcal{G}$ *to which* e *belongs. It determines a subgraph* $\mathcal{G}_k = (\mathcal{V}, \mathcal{E}_k)$ *of* $\mathcal{G}$*, where* $e \in \mathcal{E}_k$ *if and only if* $w_k(e) > 0$.

Theorem 3 $\mathbf{K}_k(\mathcal{G}) = \mathbf{K}(\mathcal{G}_k)$

PROOF: Let $u\mathbf{K}_k v$ hold in graph $\mathcal{G}$. If $u = v$, it is also true that $u\mathbf{K}v$ in graph $\mathcal{G}_k$. If the vertices u and v are different, there exists (vertex) k-gonal chain in $\mathcal{G}$ from u to v. Each edge in this chain belongs to at least one (k)-gone, so the whole chain is in $\mathcal{G}_k$. So u and v are connected in $\mathcal{G}_k$ or with other words $u\mathbf{K}v$ in $\mathcal{G}_k$. $\mathbf{K}_k(\mathcal{G}) \subseteq \mathbf{K}(\mathcal{G}_k)$.

Let $u\mathbf{K}v$ hold in graph $\mathcal{G}_k$. Then a path exists from u to v in graph $\mathcal{G}_k$. Because $\mathcal{G}_k$ is k-gonal, each edge on this path belongs to at least one

(k)-gone, so we can construct a k-gonal chain from u to v in $\mathcal{G}_k$. Because $\mathcal{G}_k$ is a subgraph of $\mathcal{G}$, this chain is also a chain in $\mathcal{G}$, which means that $u\mathbf{K}_k v$ holds in graph $\mathcal{G}$. $\mathbf{K}(\mathcal{G}_k) \subseteq \mathbf{K}_k(\mathcal{G})$. □

The last theorem has the following practical application: To determine the equivalence classes of the relation $\mathbf{K}_k$, we can first determine its k-gonal subgraph $\mathcal{G}_k$ and find its connected components afterward.

To compute the weight of the edge e we have to count to how many (k)-gons it belongs. We are still working on development of an efficient algorithm for this task for very large sparse graphs and $k \leq 5$.

The weights w_k can be used to identify dense parts of a given graph. For example, for a selected edge e in r-clique

$$w_k(e) \geq \sum_{i=3}^{k} (r-2)(r-3)\cdots(r-i+1)$$

The *Everett's k-decomposition* [7, 8, 9] of a given undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a partition $\{\mathcal{C}_1, ..., \mathcal{C}_p, \mathcal{B}_1, ..., \mathcal{B}_q\}$ of the set of edges $\mathcal{E}$, where $\mathcal{C}_i$ are *k-gonal blocks* – edge sets of maximal k-gonally connected subgraphs, and $\mathcal{B}_j$ are *bridges* – edge sets of connected components of $\mathcal{E} \setminus \cup\mathcal{C}_i$.

A procedure to determine Everett's decomposition is as follows: First determine the k-gonal subgraph $\mathcal{G}_k$. The edge sets of its connected components are by Theorem 3 just the sets $\mathcal{C}_i$. Finally determine the bridges $\mathcal{B}_i$ – the connected components on the edge set $\mathcal{E} \setminus \cup\mathcal{C}_i$.

Note that for $i \neq j$ hold $\mathcal{V}(\mathcal{C}_i) \cap \mathcal{V}(\mathcal{C}_j) = \emptyset$ and $\mathcal{V}(\mathcal{B}_i) \cap \mathcal{V}(\mathcal{B}_j) = \emptyset$.

Definition 3 *A sequence* $(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_s)$ *of* (k)*-gons of* $\mathcal{G}$ edge k-gonally connects *a vertex* $u \in \mathcal{V}$ *with a vertex* $v \in \mathcal{V}$*, if and only if*

1. $u \in \mathcal{V}(\mathcal{C}_1)$,
2. $v \in \mathcal{V}(\mathcal{C}_s)$, *and*
3. $\mathcal{E}(\mathcal{C}_{i-1}) \cap \mathcal{E}(\mathcal{C}_i) \neq \emptyset \quad$ *for* $\quad i = 2, \ldots, s$.

Such a sequence is called an edge k-gonal chain*, see Figure 3. Vertex* $u \in \mathcal{V}$ *is* edge k-gonally connected *with vertex* $v \in \mathcal{V}$*,* $u\mathbf{L}_k v$*, if and only if* $u = v$ *or there exists an edge k-gonal chain that edge k-gonally connects vertex* u *with vertex* v*.*

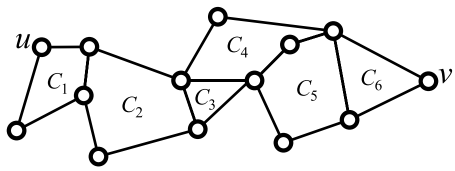

Figure 3: *Edge 5-gonal chain from u to v*

In the biconnected graph in Figure 4 the vertices u in v are edge triangularly connected, while the vertices x and z are not. The relation $\mathbf{L}_3$ is not transitive ***on vertices***: $x\mathbf{L}_3v$, $v\mathbf{L}_3z$, but not $x\mathbf{L}_3z$.

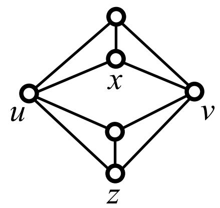

Figure 4: *Biconnected triangular graph*

Theorem 4 *The relation* $\mathbf{L}_k$ *determines an equivalence relation on the set of edges* $\mathcal{E}$.

PROOF: Let the relation $\sim$ on $\mathcal{E}$ be defined as follows: $e \sim f$, if and only if $e = f$ or there exists an edge k-gonal chain $(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_s)$, where $e \in \mathcal{E}(\mathcal{C}_1)$ and $f \in \mathcal{E}(\mathcal{C}_s)$.

Reflexivity of $\sim$ follows from its definition.

Symmetry of $\sim$ follows readily. Let e and f be two edges from $\mathcal{E}$ such that $e \sim f$. Then an edge k-gonal chain $(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_s)$ from e to f exists. The reverse $(\mathcal{C}_s, \ldots, \mathcal{C}_2, \mathcal{C}_1)$ is an edge k-gonal chain from f to e. Hence $f \sim e$.

And transitivity. Let e, f and g be edges, such that $e \sim f$ and $f \sim g$. Then there exists an edge k-gonal chain from e to f and an edge k-gonal chain from f to g. The concatenation of these two chains is an edge k-gonal chain from e to g (the (k)-gons in the contact of the chains both contain the edge f, so their intersection is not empty). Therefore $e \sim g$. □

Theorem 5 *Let* $\mathbf{B}_k = \mathbf{B} \cap \mathbf{K}_k$. *In a graph* $\mathcal{G}$ *hold:*

a. $\mathbf{K}_k \subseteq \mathbf{K}$ *b.* $\mathbf{L}_k \subseteq \mathbf{B}_k$

and for $i < j$ also:

c. $\mathbf{K}_i \subseteq \mathbf{K}_j$ *e.* $\mathbf{B}_i \subseteq \mathbf{B}_j$

d. $\mathbf{L}_i \subseteq \mathbf{L}_j$

Proof: Most properties are simple consequences of their definitions. Let us prove the property b.

Let u and v be vertices, such that $u\mathbf{L}_k v$. If $u = v$, it is also $u\mathbf{B}v$ and $u\mathbf{K}_k v$ by definition, from which it follows that $u\mathbf{B}_k v$. If the vertices are different, there exists an edge k-gonal chain from u to v. But since each edge k-gonal chain is also a vertex k-gonal chain (if two (k)-gons have a common edge, they also have a common vertex), $u\mathbf{K}_k v$ holds. The subgraph in the form of an edge k-gonal chain is biconnected [3], $u\mathbf{B}v$. Therefore $u\mathbf{B}_k v$. □

The relationships from theorem 5 can be presented by a diagram:

$$\begin{array}{ccccc} & & \mathbf{B} & \subseteq & \mathbf{K} \\ & & \cup\!\!| & & \cup\!\!| \\ \vdots & & \vdots & & \vdots \\ \cup\!\!| & & \cup\!\!| & & \cup\!\!| \\ \mathbf{L}_k & \subseteq & \mathbf{B}_k & \subseteq & \mathbf{K}_k \\ \cup\!\!| & & \cup\!\!| & & \cup\!\!| \\ \mathbf{L}_{k-1} & \subseteq & \mathbf{B}_{k-1} & \subseteq & \mathbf{K}_{k-1} \\ \cup\!\!| & & \cup\!\!| & & \cup\!\!| \\ \vdots & & \vdots & & \vdots \end{array}$$

3 Cyclic k-gonal connectivity in directed graphs

Let $\mathcal{G} = (\mathcal{V}, \mathcal{A})$ be a simple directed graph. We shall give special attention to two special types of Everett's semicycles [7, 8], see Figure 5, related to the selected *base* arc $a(u, v) \in \mathcal{A}$: *cycles* (an arc with a feed-back path) and *transitive semicycles* (an arc with a reinforcement path) of length at most k. The selected arc a of transitive semicycle is called a *transitive* arc.

For cyclic (k)-gons we define (similarly as for undirected graphs):

Definition 4 *A sequence $(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_s)$ of cycles of length at most k and at least 2 of $\mathcal{G}$ (*vertex*) *cyclic k-gonally connects* a vertex $u \in \mathcal{V}$ with a vertex $v \in \mathcal{V}$, if and only if*

1. $u \in \mathcal{V}(\mathcal{C}_1)$,

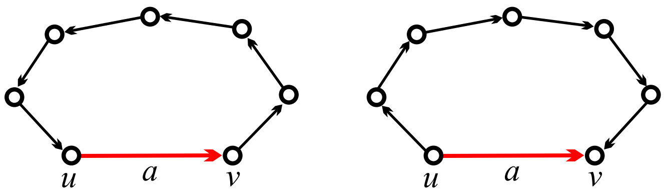

Figure 5: *Semicycles on an arc*

2. $v \in \mathcal{V}(\mathcal{C}_s)$, *and*

3. $\mathcal{V}(\mathcal{C}_{i-1}) \cap \mathcal{V}(\mathcal{C}_i) \neq \emptyset \quad$ *for* $\quad i = 2, \ldots, s$.

*Such a sequence is called a (*vertex*)* cyclic k-gonal chain. *Vertex $u \in \mathcal{V}$ is (*vertex*)* cyclic k-gonally connected *with vertex $v \in \mathcal{V}$, $u\mathbf{C}_k v$, if and only if $u = v$ or there exists a (vertex) cyclic k-gonal chain that (vertex) cyclic k-gonally connects vertex u with vertex v.*

Definition 5 *A sequence $(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_s)$ of cycles of length at most k and at least 2 of $\mathcal{G}$* arc cyclic k-gonally connects *a vertex $u \in \mathcal{V}$ with a vertex $v \in \mathcal{V}$, if and only if*

1. $u \in \mathcal{V}(\mathcal{C}_1)$,

2. $v \in \mathcal{V}(\mathcal{C}_s)$, *and*

3. $\mathcal{A}(\mathcal{C}_{i-1}) \cap \mathcal{A}(\mathcal{C}_i) \neq \emptyset \quad$ *for* $\quad i = 2, \ldots, s$.

Such a sequence is called an arc cyclic k-gonal chain. *Vertex $u \in \mathcal{V}$ is* arc cyclic k-gonally connected *with vertex $v \in \mathcal{V}$, $u\mathbf{D}_k v$, if and only if $u = v$ or there exists an arc cyclic k-gonal chain that arc cyclic k-gonally connects vertex u with vertex v.*

Between $\mathbf{C}_k$ and $\mathbf{D}_k$ similar relations hold as for $\mathbf{K}_k$ and $\mathbf{L}_k$.

Theorem 6 *A weakly connected cyclic k-gonal graph is also strongly connected.*

Proof: Take any pair of vertices u and v. Since $\mathcal{G}$ is weakly connected there exists a semipath connecting u and v. Each arc on this semipath belongs to at least one (k)-cycle. Therefore its end-points are connected by a path in opposite direction – we can construct a walk from u to v and also a walk from v to u. □

Theorem 7 *The cyclic k-gonal connectivity $\mathbf{C}_k$ is an equivalence relation on the set of vertices $\mathcal{V}$.*

Proof: The proof is similar to the proof of Theorem 1. □

An arc is *cyclic* if and only if it belongs to some cycle (of any length) in the graph $\mathcal{G} = (\mathcal{V}, \mathcal{A})$. The cyclic arcs that do not belong to some (k)-cycle are called *k-long* (range) arcs [11].

Theorem 8 *If the graph $\mathcal{G} = (\mathcal{V}, \mathcal{A})$ does not contain k-long arcs then its cyclic k-gonal reduction $\mathcal{G}/\mathbf{C}_k = (\mathcal{V}/\mathbf{C}_k, \mathcal{A}^*)$, where for $X, Y \in \mathcal{V}/\mathbf{C}_k$: $(X, Y) \in \mathcal{A}^* \iff \exists u \in X \exists v \in Y : (u, v) \in \mathcal{A}$, is an acyclic graph.*

Proof: Suppose that cyclic k-gonal reduction of graph $\mathcal{G}$ is not acyclic. Then it contains a cycle C^*, which can be extended to a cycle C of graph $\mathcal{G}$. Let a^* be any arc of C^* and let a be the corresponding arc of C. Because the end-points of a^* are different, the end-points of a belong to two different components of the relation $\mathbf{C}_k$. So a does not belong to any cyclic (k)-gone. But a is cyclic (it belongs to cycle C), so it is a k-long arc. This is a contradiction. Therefore, the cyclic k-gonal reduction of graph $\mathcal{G}$ must be acyclic. □

This theorem tells us that the 'global structure' of a graph without k-long arcs is essentially acyclic – hierarchical. From this proof we also see how to identify the k-long arcs. They are exactly the arcs that are reduced to cyclic arcs in $\mathcal{G}/\mathbf{C}_k$.

Theorem 9 *The relation $\mathbf{D}_k$ determines an equivalence relation on the set of arcs $\mathcal{A}$.*

Proof: The proof is similar to the proof of Theorem 4. □

Definition 6 *The vertices $u, v \in \mathcal{V}$ are (*vertex*)* strongly k-gonally connected, *$u\mathbf{S}_k v$, if and only if $u = v$ or there exists strongly connected k-gonal subgraph that contains u and v.*

It is easy to see that $\mathbf{D}_k \subseteq \mathbf{C}_k \subseteq \mathbf{S}_k$. The relationships between these relations can be presented by the following diagram, where $\mathbf{S}$ is the strong connectivity relation.

$$\begin{array}{ccccc} & & & & \mathbf{S} \\ & & & & \cup\!\vert \\ \vdots & & \vdots & & \vdots \\ \cup\!\vert & & \cup\!\vert & & \cup\!\vert \\ \mathbf{D}_k & \subseteq & \mathbf{C}_k & \subseteq & \mathbf{S}_k \\ \cup\!\vert & & \cup\!\vert & & \cup\!\vert \\ \mathbf{D}_{k-1} & \subseteq & \mathbf{C}_{k-1} & \subseteq & \mathbf{S}_{k-1} \\ \cup\!\vert & & \cup\!\vert & & \cup\!\vert \\ \vdots & & \vdots & & \vdots \end{array}$$

We can define three networks that can provide us with more detailed picture about the graph structure:

- *Feedback network* $\mathcal{N}_F = (\mathcal{V}, \mathcal{A}, w_F)$ where $w_F(a)$ is the number of different (k)-cycles containing the arc a.
- *Transitive network* $\mathcal{N}_T = (\mathcal{V}, \mathcal{A}, w_T)$ where $w_T(a)$ is the number of different transitive (k)-semicycles containing the arc a as the transitive arc (shortcut).
- *Support network* $\mathcal{N}_S = (\mathcal{V}, \mathcal{A}, w_S)$ where $w_S(a)$ is the number of different transitive (k)-semicycles containing the arc a as a nontransitive arc.

Theorem 10 $\mathbf{C}_k(\mathcal{G}) = \mathbf{S}(\mathcal{G}_F)$, *where* $\mathcal{G}_F = (\mathcal{V}, \mathcal{A}_F)$ *and* $\mathcal{A}_F = \{a \in \mathcal{A} : w_F(a) > 0\}$.

PROOF: Let $u\mathbf{C}_k v$ hold in graph $\mathcal{G}$. If $u = v$, it is also true that $u\mathbf{S}v$ holds in graph $\mathcal{G}_F$. If the vertices are different, a cyclic k-gonal chain from u to v exists in $\mathcal{G}$. Each arc in this chain belongs to at least one (k)-cycle, so the whole chain is in $\mathcal{G}_F$. Vertices u and v are mutually reachable by arcs of this chain, so $u\mathbf{S}v$ holds in $\mathcal{G}_F$.

Let $u\mathbf{S}v$ hold in graph $\mathcal{G}_F$. Then a walk from u to v exists in graph $\mathcal{G}_F$. Because $\mathcal{G}_F$ is cyclic k-gonal, each arc on this walk belongs to at least one (k)-cycle, so we can construct a cyclic k-gonal chain from u to v in $\mathcal{G}_F$. Because $\mathcal{G}_F$ is subgraph of $\mathcal{G}$, this chain is also in $\mathcal{G}$, which means that $u\mathbf{C}_k v$ holds in graph $\mathcal{G}$. □

4 Transitivity

Let $\mathbf{R}(\mathcal{G})$ denote the *reachability* relation in a given directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{A})$. Vertex v is *reachable* from vertex u, $u\mathbf{R}v$, if and only if $u = v$ or a walk from u to v exists.

Theorem 11 *If we remove all (or some) arcs belonging to a triangularly transitive path π (all arcs of π are triangularly transitive) from a graph $\mathcal{G} = (\mathcal{V}, \mathcal{A})$ the reachability relation does not change: $\mathbf{R}(\mathcal{G}) = \mathbf{R}(\mathcal{G} \setminus \mathcal{A}(\pi))$.*

PROOF: Because the graph $\mathcal{G} \setminus \mathcal{A}(\pi)$ is a subgraph of $\mathcal{G}$, it is obvious that $\mathbf{R}(\mathcal{G} \setminus \mathcal{A}(\pi)) \subseteq \mathbf{R}(\mathcal{G})$. To prove the converse let a be any arc on the triangularly transitive path π. Because of transitivity of the arc a, its terminal vertex is also reachable from its initial vertex by two supporting arcs. We only have to check that none of them belongs to the path π, so it is not deleted. Because the arc a and each its supporting arc have a common vertex, the only way to be on the same path is to be consecutive arcs. But this is impossible because of their directions. This means that any walk on $\mathcal{G}$ can be transformed into a walk on $\mathcal{G} \setminus \mathcal{A}(\pi)$ by replacing arcs belonging to $\mathcal{A}(\pi)$ with the corresponding supporting pairs. Therefore also the converse is true: $\mathbf{R}(\mathcal{G}) \subseteq \mathbf{R}(\mathcal{G} \setminus \mathcal{A}(\pi))$. □

But we cannot remove all triangularly transitive arcs. The counter-example is presented in Figure 6, where we have a directed 6-cycle whose vertices are connected by arcs with additional vertex in its center. The central vertex is reachable from anywhere. All the arcs from the cycle to the central vertex are transitive. If we remove them all, the central vertex is not reachable any more.

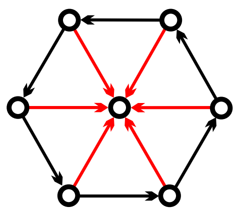

Figure 6: *Graph in which all triangularly transitive arcs cannot be removed*

The theorem also cannot be generalized to arbitrary transitive paths. The counter-example in Figure 7 presents a graph with a transitive path

$u - x - y - v$. If we remove this path, vertex v is not reachable from vertex u any more.

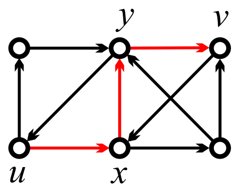

Figure 7: *Graph in which a transitive path cannot be removed*

Let $\mathbf{T}_k$ denote the k-transitive reachability relation in a given directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{A})$.

Definition 7 *Vertex v is k-*transitively reachable *from vertex u, $u\mathbf{T}_k v$, if and only if $u = v$ or a walk from u to v exists in which each arc is k-transitive – it is a base (shortcut arc) of some transitive semicycle of length at most k.*

The vertices u and v are mutually k-transitively reachable, if vertex u is k-transitively reachable from vertex v and vertex v is k-transitively reachable from vertex u. We denote this relation by $\hat{\mathbf{T}}_k$

$$u\hat{\mathbf{T}}_k v \Leftrightarrow u\mathbf{T}_k v \wedge v\mathbf{T}_k u$$

Theorem 12 *The relation of mutual k-transitive reachability $\hat{\mathbf{T}}_k = \mathbf{T}_k \cap \mathbf{T}_k^{-1}$ is an equivalence relation on the set of vertices V.*

PROOF: It is well known that if $\mathbf{Q}$ is a reflexive and transitive relation then $\hat{\mathbf{Q}} = \mathbf{Q} \cap \mathbf{Q}^{-1}$ is an equivalence relation. The relation $\mathbf{T}_k$ is reflexive by definition, so we have only to prove that it is also transitive.

Let u, v and w be such vertices that $u\mathbf{T}_k v$ and $v\mathbf{T}_k w$. If these vertices are not pairwise different, the transitivity condition is trivially true. Otherwise a walk from u to v and a walk from v to w exist, in which every arc is k-transitive. Their concatenation is a walk from u to w, in which every arc is k-transitive, so $u\mathbf{T}_k w$ holds. □

5 Further generalizations

Till now we were considering the connectivity by triangles and other short cycles. Intersections of two consecutive cycles in the corresponding chains contained at least one vertex (vertex connectivity) or at least one edge/arc (edge/arc connectivity). This can be generalized to other families of graphs.

Definition 8 *Let* $\mathbb{H}$ *and* $\mathbb{H}_0$ *be two families of graphs. A sequence* $(\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_s)$ *of subgraphs of* $\mathcal{G}$ $(\mathbb{H}, \mathbb{H}_0)$ connects *a vertex* $u \in \mathcal{V}$ *with a vertex* $v \in \mathcal{V}$*, if and only if*

1. $u \in \mathcal{V}(\mathcal{H}_1)$,
2. $v \in \mathcal{V}(\mathcal{H}_s)$,
3. $\mathcal{H}_i \in \mathbb{H} \quad$ *for* $\quad i = 1, \ldots, s$, *and*
4. $\mathcal{H}_{i-1} \cap \mathcal{H}_i \supseteq \mathcal{H} \in \mathbb{H}_0 \quad$ *for* $\quad i = 2, \ldots, s$.

Example: For $r < k$ we can define (k, r)-clique connectivity: $\mathbb{H} = \{K_{r+1}, K_{r+2}, \ldots, K_k\}$, $\mathbb{H}_0 = \{K_r\}$

All the types of connectivity introduced in this paper are special cases of the generalized connectivity:

$$\begin{aligned}
\mathbf{K}_k &= (\{C_3, \ldots, C_k\}, \{K_1\}) \text{ connectivity} \\
\mathbf{L}_k &= (\{C_3, \ldots, C_k\}, \{K_2\}) \text{ connectivity}
\end{aligned}$$

For the generalized connectivity similar theorems hold as for k-gonal connectivity.

6 Conclusion

In the paper we introduced different kinds of short cycle connectivities and proved their basic properties. The corresponding networks provide us with a powerful tool for identification of dense parts of graph with applications in the design of algorithms and in social network analysis.

The support for triangular connectivities and networks is provided in Pajek – program for analysis and visualization of large networks [2].

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