Notes on Blockmodeling*

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Abstract

We attempt to develop further the blockmodeling of networks, so as better to capture the network structure. For this purpose a richer structure than ordinary (valued) graphs has to be used for a model. Such structures are valued graphs with typified (complete, dominant, regular, ...) connections. Based on the proposed formalization, the blockmodeling is cast as an optimization problem.

Keywords: blockmodels, types of connection, averaging rules, optimization.

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1 Introduction

The paper is an elaboration of the following two basic observations:

- in blockmodeling we have two basic subproblems:
  - partitioning of units – determining the classes (clusters) that form the vertices in a model;

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- determining the links in a model (and their values);

- description of a model as a (valued) graph is often unprecise. For example, how does one denote the father–sons connection? A richer structure is needed for a model to be able to describe the network structure properly.

In this paper we propose a generalization of blockmodeling which enables us to capture better the network structure. It unifies and combines different notions of equivalences (structural, regular, ...), which can be simultaneously applied to the same network. The paper deals mainly with formal and computational aspects of the proposed approach. A discussion of its methodological aspects and its relations to standard blockmodeling methods is given in Doreian, Batagelj and Ferligoj (1994).

2 Graphs and networks

A *graph* is an ordered triple $G = (V, E, A)$ where V, E and A are pairwise disjoint sets. The set V is a *vertex-set* of graph G; E is the set of *edges* (undirected lines), and A is the set of *arcs* (directed lines) of graph G. Sets E and A can also be empty. If $A = \emptyset$, the graph G is *undirected*; and it is *directed* if $E = \emptyset$. To each *line* from $L = E \cup A$ belongs a pair of vertices – its *ends*. In the case of an arc one vertex is its *initial* vertex, and the other vertex is its *terminal* vertex.

That the edge e has end vertices u and v we write as $e(u{:}\, v)$, or equivalently $e(v{:}\, u)$. Similarly, $a(u, v)$ says that u is the initial and v is the terminal vertex of arc a. A line $p \in L$ *joins* its end vertices, and an arc $a \in A$ *joins* its initial vertex *to* its terminal vertex. When both ends of a line are equal we call it a *loop*. A vertex which is not an end of any line is called *isolated*.

We extend our notation for edges and arcs to all lines by: let $p \in L$, then

$$p(u, v) \equiv (p \in E \wedge p(u{:}\, v)) \vee (p \in A \wedge p(u, v))$$

line p joins vertex u to vertex v, and

$$p(u{:}\, v) \equiv p(u, v) \vee p(v, u)$$

line p joins vertices u and v.

We shall use the following abbreviation:

$$uLv \equiv \exists p \in L : p(u, v)$$

which essentially defines the *adjacency* relation. For $\emptyset \subset X, Y \subseteq V$ we also define a *block*

$$L(X, Y) = \{p \in L : \exists x \in X \exists y \in Y : p(x, y)\}$$

and a *complete block*

$$K(X, Y) = \{(x{:}\, y) : x \in X, y \in Y\} \cup \{(x, y) : x \in X, y \in Y\}$$

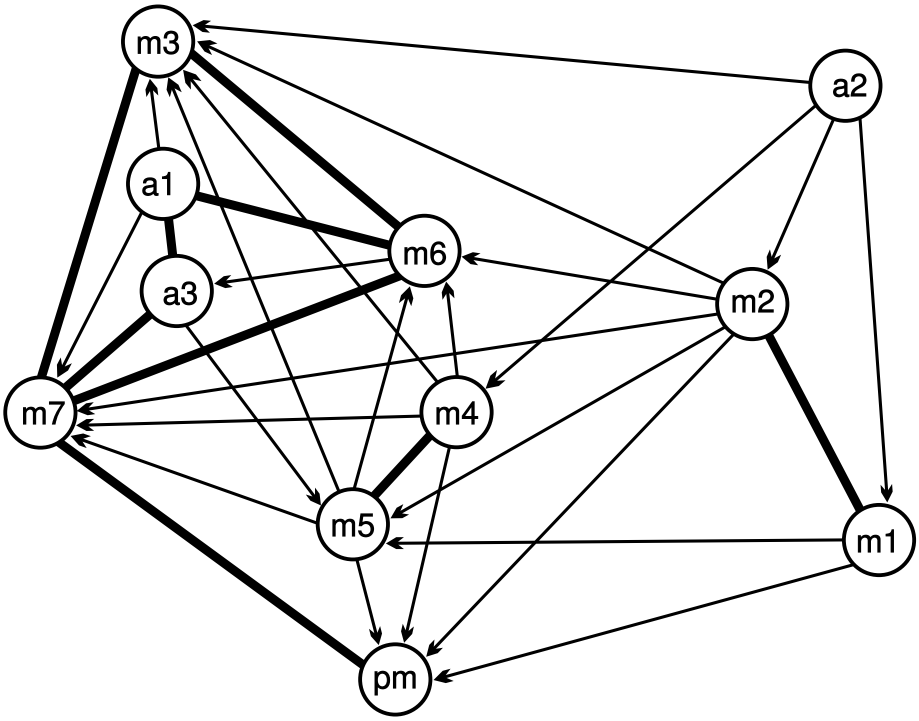

Figure 1: Network graph: Student Government – discussion, recall

A block of the form $L(X, X)$ is called a *diagonal* block, and a block of the form $L(X, Y)$, $X \cap Y = \emptyset$, an *off-diagonal* block.

A graph is *simple* iff each pair of adjacent vertices is either joined by an edge, or by an arc, or by a pair of arcs of opposite directions, or by a directed loop. In the following we shall assume all graphs directed and simple.

A (simple) *network* is an ordered triple $\mathcal{N} = (V, L, \nu)$ where

- (V, L) is a simple graph,
- $\nu : L \to S$ assigns values to lines, and S is the set of possible values.

In this context, the set of vertices V is usually called the *set of units*. When the values of lines are not given we assume the 'default' values

$$\nu(x, y) = \begin{cases} 1 & xLy \\ 0 & \text{otherwise} \end{cases}$$

An example of a network is presented on Figure 1. For a detailed description and the complete data see Hlebec (1993). The units are members of Student Government at University of Ljubljana in May 1992 (a – advisor, m – minister, pm – prime minister), and the relation is determined by the answers (based on the respondent's recall) to the question:

Table 1: Types of connection

null	$\mathrm{nul}(X, Y; L)$	$\equiv$	$L(X, Y) = \emptyset$
complete	$\mathrm{com}(X, Y; L)$	$\equiv$	$\forall x \in X \forall y \in Y : (x \neq y \Rightarrow xLy)$
row-dominant	$\mathrm{rdo}(X, Y; L)$	$\equiv$	$\exists x \in X \forall y \in Y : (x \neq y \Rightarrow xLy)$
col-dominant	$\mathrm{cdo}(X, Y; L)$	$\equiv$	$\mathrm{rdo}(Y, X; L^{-1})$
row-regular	$\mathrm{rre}(X, Y; L)$	$\equiv$	$\forall x \in X \exists y \in Y : xLy$
col-regular	$\mathrm{cre}(X, Y; L)$	$\equiv$	$\mathrm{rre}(Y, X; L^{-1})$
regular	$\mathrm{reg}(X, Y; L)$	$\equiv$	$\mathrm{cre}(X, Y; L) \wedge \mathrm{rre}(X, Y; L)$
row-functional	$\mathrm{rfn}(X, Y; L)$	$\equiv$	$\forall y \in Y \exists! x \in X : xLy$
col-functional	$\mathrm{cfn}(X, Y; L)$	$\equiv$	$\forall x \in X \exists! y \in Y : xLy$

"Of the members and advisors of the Student Government, whom do you (most often) informally talk with?"

3 Types of connection

Two sets of vertices $X, Y \subseteq V$ can be related in different ways. We describe these *types* of connection by predicates, where X is considered as the ego-set (see Table 1 and Figure 2).

Let T denote a connection type predicate. These predicates have several characteristic properties

property	T
$T(X_1, Y_1) \wedge T(X_2, Y_2) \Rightarrow T(X_1 \cup X_2, Y_1 \cup Y_2)$	reg, rre, cre
$T(X_1, Y) \wedge T(X_2, Y) \Rightarrow T(X_1 \cup X_2, Y)$	com, rfn, nul, (reg, rre, cre)
$T(X, Y_1) \wedge T(X, Y_2) \Rightarrow T(X, Y_1 \cup Y_2)$	com, cfn, nul, (reg, rre, cre)
$\emptyset \subset Z \subseteq X \wedge T(X, Y) \Rightarrow T(Z, Y)$	com, nul, cdo, rfn, rre
$\emptyset \subset Z \subseteq Y \wedge T(X, Y) \Rightarrow T(X, Z)$	com, nul, rdo, cfn, cre
$T(X, Y) \Rightarrow T(X \cup Z, Y)$	rdo, cre
$T(X, Y) \Rightarrow T(X, Y \cup Z)$	cdo, rre

and several relations hold among them:

$$\begin{array}{cc} \mathrm{reg}(X,Y) \Rightarrow \mathrm{cre}(X,Y), & \mathrm{reg}(X,Y) \Rightarrow \mathrm{rre}(X,Y), \\ \mathrm{com}(X,Y) \Rightarrow \mathrm{rdo}(X,Y), & \mathrm{rfn}(X,Y) \Rightarrow \mathrm{rre}(X,Y), \\ \mathrm{com}(X,Y) \Rightarrow \mathrm{cdo}(X,Y), & \mathrm{cfn}(X,Y) \Rightarrow \mathrm{cre}(X,Y), \\ \multicolumn{2}{c}{X \cap Y = \emptyset \wedge \mathrm{rdo}(X,Y) \Rightarrow \mathrm{cre}(X,Y),} \\ \multicolumn{2}{c}{X \cap Y = \emptyset \wedge \mathrm{cdo}(X,Y) \Rightarrow \mathrm{rre}(X,Y).} \end{array}$$

Often a selected type of connection is restricted to diagonal/off-diagonal blocks.

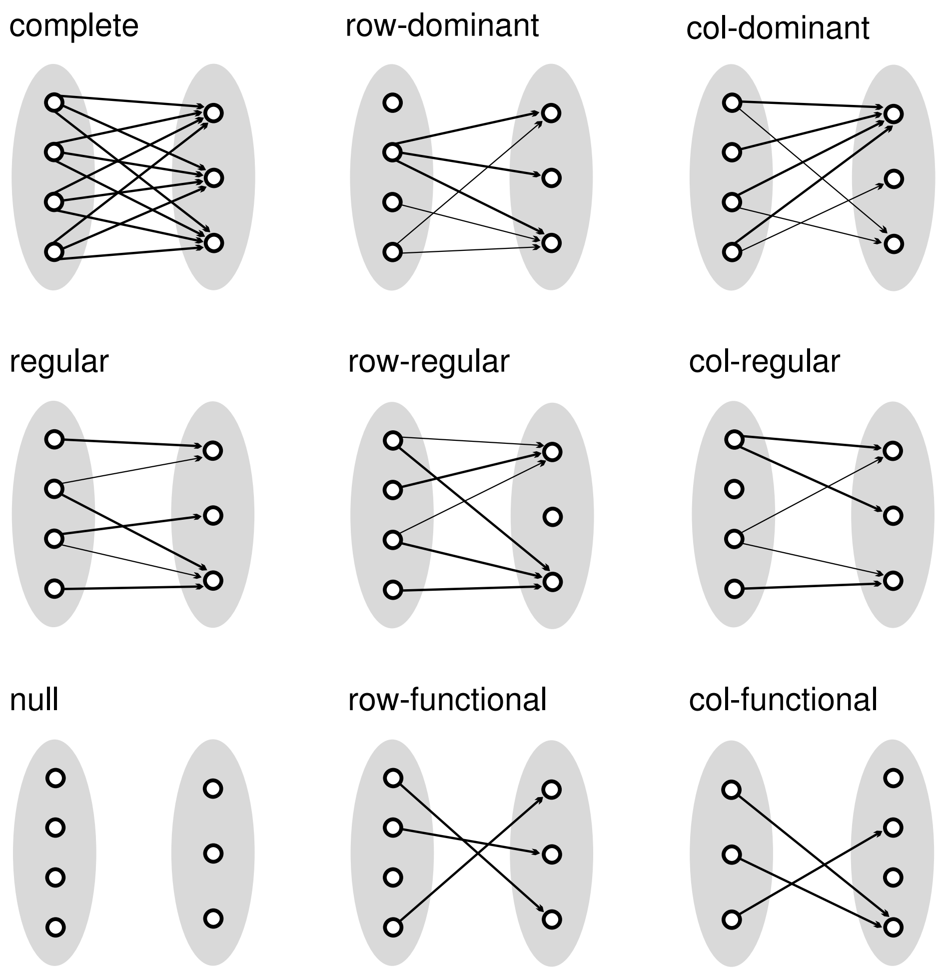

Figure 2: Types of connection between two sets; the left set is the ego-set.

Another group of predicates is based on the notion of vertex degree. Examples of such predicates are:

degree density γ	$\text{den}(\gamma)(X, Y; L)$	$\equiv$	$\text{card}\, L(X, Y) \geq \gamma\, \text{card}(X \times Y)$
degree bound n	$\text{deg}(n)(X, Y; L)$	$\equiv$	$\forall x \in X : \text{card}(L(x) \cap Y) \geq n$

where card A denotes the cardinality – number of elements in the set A.

More complicated predicates expressing partial ordering, different types of connectivity, ..., simultaneous consideration of $X \times Y$ and $Y \times X$, and even n-ary, $n > 2$ predicates could also be considered. In some applications a *don't care* predicate, which is always satisfied (true), can be useful. In this paper we shall limit our discussion to binary predicates.

In the definition of predicates we can also consider values of lines in the block. For example, for searching balanced/clusterable partitions of a network two predicates are needed (Doreian and Mrvar 1994):

positive	$\text{pos}(X, Y; L)$	$\equiv$	$\forall x \in X, y \in Y : (xLy \Rightarrow \nu(x, y) > 0)$
negative	$\text{neg}(X, Y; L)$	$\equiv$	$\forall x \in X, y \in Y : (xLy \Rightarrow \nu(x, y) < 0)$

4 Blockmodeling

4.1 Blockmodels

A *blockmodel* is an ordered sextuple $\mathcal{M} = (U, K, \mathcal{T}, Q, \pi, \alpha)$ where:

- U is a set of *types* of units (images or representatives of classes);
- $K \subseteq U \times U$ is a set of *connections*;
- $\mathcal{T}$ is a set of predicates used to describe the types of connections between different classes (clusters, groups, types of units) in a network. We assume that nul $\in \mathcal{T}$. A mapping $\pi : K \to \mathcal{T} \setminus \{\text{nul}\}$ assigns predicates to connections;
- Q is a set of *averaging rules.* A mapping $\alpha : K \to Q$ determines rules for computing values of connections.

Let us denote by $\mu : V \to U$ a mapping which maps classes of units to the corresponding types. Then we define for $t \in U$

$$C(t) = \mu^{-1}(t) = \{x \in V : \mu(x) = t\}.$$

Therefore

$$\mathcal{C}(\mu) = \{C(t) : t \in U\}$$

is a partition (clustering) of the set of units V.

A (surjective) mapping $\mu : V \to U$ determines a blockmodel $\mathcal{M}$ of network $\mathcal{N}$ iff it satisfies the conditions:

$$\forall (t, w) \in K : \pi(t, w)(C(t), C(w))$$

and

$$\forall (t, w) \in U \times U \setminus K : \mathrm{nul}(C(t), C(w)).$$

Note that, if we set $\mathcal{T} = \{\mathrm{nul}, \mathrm{com}\}$ we are asking for a structural blockmodel (Lorrain and White 1971); and, if we set $\mathcal{T} = \{\mathrm{nul}, \mathrm{reg}\}$ we are asking for a regular blockmodel (White and Reitz 1983).

4.2 Equivalences

Let $\approx$ be an equivalence relation over V. It partitions the set of units V into classes (clusters)

$$[x] = \{y \in V : x \approx y\}.$$

We say that $\approx$ is *compatible* with $\mathcal{T}$ over a network $\mathcal{N}$ iff

$$\forall x, y \in V \exists T \in \mathcal{T} : T([x], [y]).$$

It is easy to verify that the notion of compatibility for $\mathcal{T} = \{\mathrm{nul}, \mathrm{reg}\}$ reduces to the usual definition of regular equivalence (Borgatti and Everett 1989).

For a compatible equivalence $\approx$ the mapping $\mu : x \mapsto [x]$ determines a blockmodel.

4.3 Averaging rules

The next question is how to determine the values of connections in a way compatible with their types and values of corresponding lines in a network? This problem can be approached by selecting/determining an appropriate *averaging rule*.

For $t, w \in U$, let $X = C(t)$ and $Y = C(w)$; then general requirements for an averaging rule $\overline{\nu} : K \to S$ could be

$$\mathrm{nul}(X, Y; L) \Rightarrow \overline{\nu}(t, w) = 0,$$

and

$$(\forall p \in L(X, Y) : \nu(p) = c) \quad \Rightarrow \quad \overline{\nu}(t, w) = c,$$

or

$$\sum_{p \in L(X,Y)} \nu(p) = N(t, w) \overline{\nu}(t, w),$$

where $N(t, w)$ is the *multiplicity* of connection (t, w). The multiplicity $N(t, w)$ depends also on the corresponding type of connection. For example, we can set:

$$\begin{aligned} \mathrm{com}(X, Y; L) &\Rightarrow N(t, w) = \mathrm{card}(X \times Y), \\ \mathrm{rre}(X, Y; L) &\Rightarrow N(t, w) = \mathrm{card}\, Y, \\ \mathrm{reg}(X, Y; L) &\Rightarrow N(t, w) = \max(\mathrm{card}\, X, \mathrm{card}\, Y). \end{aligned}$$

Table 2: Characterizations of types of blocks

null	all 0 (except may be diagonal)
complete	all 1 (except may be diagonal)
row-dominant	$\exists$ all 1 row (except may be diagonal)
col-dominant	$\exists$ all 1 column (except may be diagonal)
row-regular	1-covered rows
col-regular	1-covered columns
regular	1-covered rows and 1-covered columns
row-functional	exactly one 1 in each column
col-functional	exactly one 1 in each row
density γ	# of 1s $\geq \gamma \cdot size$

There are several examples of such averaging rules:

$$\begin{aligned}
\text{ave}(X,Y) &= \frac{1}{\text{card}\, L(X,Y)} \sum_{p \in L(X,Y)} \nu(p) \\
\text{row-ave}(X,Y) &= \frac{1}{\text{card}\, X} \sum_{p \in L(X,Y)} \nu(p) \\
\max(X,Y) &= \max_{p \in L(X,Y)} \nu(p) \\
\text{med}(X,Y) &= \text{med}_{p \in L(X,Y)} \nu(p)
\end{aligned}$$

where med is the median operation.

These notions can be naturally generalized to multiple networks $\mathcal{N} = (V, \{L_i\}, \{\nu_i\})$ where (V, L_i) are graphs and $\nu_i : L_i \to S_i$ values of lines.

5 Optimization

To cast blockmodeling problem as an optimization problem, we can use the approach presented in Batagelj, Doreain and Ferligoj (1992) and Batagelj, Ferligoj and Doreian (1992).

Given a set of types of connection $\mathcal{T}$ and a block $L(X,Y)$, we can determine the strongest (according to the ordering of the set $\mathcal{T}$) type T which is satisfied by $L(X,Y)$. In this case we set

$$\pi(\mu(X), \mu(Y)) = T$$

But what is to be done, if no type from $\mathcal{T}$ is satisfied?

We can introduce the set of *ideal blocks* for a given type $T \in \mathcal{T}$

$$\mathcal{B}(X,Y;T) = \{B \subseteq K(X,Y) : T(B)\}$$

Table 3: Deviation measures for types of blocks

	$\delta(X, Y; T)$	
null	s_t	off-diagonal
	$s_t + d - s_d$	diagonal
complete	$n_r n_c - s_t$	off-diagonal
	$n_r n_c - s_t + d + s_d - n_r$	diagonal
row-dominant	$(n_c - m_r - 1)n_r$	diagonal, $s_d = 0$
	$(n_c - m_r)n_r$	otherwise
col-dominant	$(n_r - m_c - 1)n_c$	diagonal, $s_d = 0$
	$(n_r - m_c)n_c$	otherwise
row-regular	$(n_r - p_r)n_c$	
col-regular	$(n_c - p_c)n_r$	
regular	$(n_c - p_c)n_r + (n_r - p_r)p_c$	
row-functional	$s_t - p_r + (n_r - p_r)n_c$	
col-functional	$s_t - p_c + (n_c - p_c)n_r$	
density γ	$\max(0, \gamma n_r n_c - s_t)$	

and define the *deviation* $\delta(X, Y; T)$ of a block $L(X, Y)$ from the nearest ideal block. In Table 2 it is shown that for types from Table 1, we can efficiently test whether the block $L(X, Y)$ is of the type T. On the basis of these characterizations we can also construct the corresponding measures of deviation (see Table 3) from the ideal realization. The quantities used in the expressions for deviations have the following meaning:

s_t	– total block sum = # of 1's in a block,
s_d	– diagonal block sum = # of 1's on a diagonal,
d	– diagonal error = $\min(s_d, n_r - s_d)$,
n_r	– # of rows in a block = card X,
n_c	– # of columns in a block = card Y,
p_r	– # of non-null rows in a block,
p_c	– # of non-null columns in a block,
m_r	– maximal row-sum,
m_c	– maximal column-sum.

Note that all deviations from Table 3 are *sensitive*

$$\delta(X, Y; T) = 0 \Leftrightarrow T(L(X, Y)).$$

Therefore a block $L(X, Y)$ is of a type T exactly when the corresponding deviation is $\delta(X, Y; T)$ is 0.

In deviation δ we can also incorporate values of lines ν.

Based on deviation $\delta(X, Y; T)$ we introduce the *block-error* $\varepsilon(X, Y; T)$ of $L(X, Y)$ for type T. Two examples of block-errors are

$$\varepsilon_1(X, Y; T) = w(T)\delta(X, Y; T)$$

and

$$\varepsilon_2(X, Y; T) = \frac{w(T)}{n_r n_c}(1 + \delta(X, Y; T)),$$

where $w(T) > 0$ is a weight of type T.

We extend the block-error to the set of feasible types $\mathcal{T}$ by defining

$$\varepsilon(X, Y; \mathcal{T}) = \min_{T \in \mathcal{T}} \varepsilon(X, Y; T)$$

and

$$\pi(\mu(X), \mu(Y)) = \operatorname*{argmin}_{T \in \mathcal{T}} \varepsilon(X, Y; T)$$

To make π well-defined, we order (priorities) the set $\mathcal{T}$ and select the first type from $\mathcal{T}$ which minimizes ε.

We combine block-errors into a *total error* – blockmodeling *criterion function*

$$P(\mu; \mathcal{T}) = \sum_{(t,w) \in U \times U} \varepsilon(\mu^{-1}(t), \mu^{-1}(w); \mathcal{T}).$$

The criterion functions based on block-errors ε_1 and ε_2 are denoted P_1 and P_2 respectively.

For criterion function $P_1(\mu)$ we have

$$P_1(\mu) = 0 \quad \Leftrightarrow \quad \mu \text{ is an exact blockmodeling}$$

Also for P_2, we obtain an exact blockmodeling μ iff the deviations of all blocks are 0.

The obtained optimization problem can be solved by local optimization (Batagelj 1991; Batagelj, Doreian and Ferligoj 1992).

Once a partitioning μ and types of connection π are determined, we can also compute the values of connections. Examples of *averaging rules* for *interval* and *ordinal* networks are proposed in Table 4 where

$$\nu_t = \sum_{x \in X, y \in Y} \nu(x, y).$$

5.1 Benefits from the optimization approach to blockmodeling

In the context of optimization approach, several questions can be considered, concerning the blockmodeling problem:

- *ordinary blockmodeling*: Given a network $\mathcal{N}$ and set of types of connection $\mathcal{T}$, determine $\mathcal{M}$, i.e., μ, π and α;

Table 4: Averaging rules for types of blocks

	interval scale		ordinal scale
null	$\nu_t/(n_r n_c)$		0
complete	$\nu_t/(n_r n_c)$	other	med $\nu[X,Y]$
	$\nu_t/(n_r n_c - n_r)$	diag, $d=0$	
row-dominant	ν_t/n_c		med $\nu[r,Y]$
col-dominant	ν_t/n_r		med $\nu[X,c]$
row-regular	ν_t/n_r		med $\nu[\max X, Y]$
col-regular	ν_t/n_c		med $\nu[X, \max Y]$
regular	$\nu_t/\max(n_r, n_c)$		min(med $\nu[\max X, Y]$, med $\nu[X, \max Y]$)
row-functional	ν_t/n_r		med $\nu[\max X, Y]$
col-functional	ν_t/n_c		med $\nu[X, \max Y]$
density γ	$\gamma\nu_t/(n_r n_c)$	other	med upper($\gamma, \nu[X,Y]$)
	$\gamma\nu_t/(n_r n_c - n_r)$	diag, $d=0$	

- *evaluation of the quality of a model, comparing different models, analyzing the evolution of a network* (Sampson data, Doreian and Mrvar 1994): Given a network $\mathcal{N}$, a model $\mathcal{M}$, and blockmodeling μ, compute the corresponding criterion function;
- *model fitting*: Given a network $\mathcal{N}$, set of types $\mathcal{T}$, and a model $\mathcal{M}$, determine μ which minimizes the criterion function.

There are other possibilities:

- we can fit the network to a partial model and analyze the residual afterward;
- we can also introduce different constraints on the model, for example: units x and y are of the same type; or, types of units x and y are not connected; ...

6 Example

As an example we present some blockmodels for a Student Government Discussion network.

For a criterion function, we selected $P \equiv P_1$ with all weights equal to 1. We also excluded trivial (row,col-)dominant blocks.

For each problem, determined by $P(\mu; \mathcal{T}, k)$, we performed 200 runs of local optimization. Moving a unit from one cluster to another cluster and interchanging of two units from different clusters were used as local transformations. The results are presented in Table 5, where rows correspond to the number k of classes in partitions, and columns to selected types of connection $\mathcal{T}$. Entries in the table contain the corresponding minimal values of

Table 5: Values of optimal partitions

k	str	reg	com, reg rdo, cdo	rdo, cdo	cdo	rdo	cdo(dia) reg(dia)
2	29	4	1/2	1/1	1/1	11	2
3	23	7	0/4	0/2	2/2	5	3
4	21	7	0/3	0/1	4/3	3	4
5	15	6	1/5	2/14	4/1	3	7

Table 6: Optimal partitions, $\mathcal{T} = \{$ nul, com, rdo, cdo, reg $\}$

	partition	P
$\mathcal{C}^a_{2,1}$	{m1, pm, m2, m3, m5, m6, m7, a1, a3} {m4, a2}	1
$\mathcal{C}^a_{2,2}$	{m1, a2} {pm, m2, m3, m4, m5, m6, m7, a1, a3}	1
$\mathcal{C}^a_{3,1}$	{m1, pm, m2, m3, m4, m5, m7} {m6, a3} {a1, a2}	0
$\mathcal{C}^a_{3,2}$	{m1, m2, a2} {pm, m3, m4, m5, m6, m7} {a1, a3}	0
$\mathcal{C}^a_{3,3}$	{m1, m2} {pm, a3} {m3, m4, m5, m6, m7, a1, a2}	0
$\mathcal{C}^a_{3,4}$	{m1, m4} {pm, a3} {m2, m3, m5, m6, m7, a1, a2}	0
$\mathcal{C}^a_{4,1}$	{m1, m2} {pm, m4} {m3, m5, m6, m7, a2} {a1, a3}	0
$\mathcal{C}^a_{4,2}$	{m1, m2, a2} {pm, m4} {m3, m5, m6, m7} {a1, a3}	0
$\mathcal{C}^a_{4,3}$	{m1, m2, a2} {pm, m4, m6, m7} {m3, m5} {a1, a3}	0
$\mathcal{C}^a_{5,1}$	{m1, m2} {pm, m3} {m4, a3} {m5, a1, a2} {m6, m7}	1
$\mathcal{C}^a_{5,2}$	{m1, m2, a2} {pm, m4} {m3, m5} {m6, m7} {a1, a3}	1
$\mathcal{C}^a_{5,3}$	{m1, m2, a2} {pm, m4} {m3, m6} {m5, m7} {a1, a3}	1
$\mathcal{C}^a_{5,4}$	{m1, a2} {pm, m3} {m2, a3} {m4, m5} {m6, m7, a1}	1
$\mathcal{C}^a_{5,5}$	{m1, a3} {pm, m5} {m2, m7, a1} {m3, m4} {m6, a2}	1

criterion function P; and as a second number, if present, the number of different optimal partitions, which are listed in Tables 6, 7 and 8.

For an example of detailed presentation, we selected the solution $\mathcal{C}^a_{4,2}$ from Table 6. It is presented by a picture in Figure 3, and by a matrix rearranged by classes, in Table 9. The numbers of units in the table refer to the original ordering of units in Hlebec (1993).

The corresponding model matrix and value matrix are given in Table 10. The model is graphically represented in Figure 4.

Table 7: Optimal partitions, $\mathcal{T}$ = { nul, rdo, cdo }

	partition	P
$\mathcal{C}^d_{2,1}$	{m1, a2} {pm, m2, m3, m4, m5, m6, m7, a1, a3}	1
$\mathcal{C}^d_{3,1}$	{m1, m4} {pm, a3} {m2, m3, m5, m6, m7, a1, a2}	0
$\mathcal{C}^d_{3,2}$	{m1, m2} {pm, a3} {m3, m4, m5, m6, m7, a1, a2}	0
$\mathcal{C}^d_{4,1}$	{m1, m2} {pm, m4} {m3, m5, m6, m7, a2} {a1, a3}	0

Table 8: Optimal partitions, $\mathcal{T}$ = { nul, cdo }

	partition	P
$\mathcal{C}^c_{2,1}$	{m1, a2} {pm, m2, m3, m4, m5, m6, m7, a1, a3}	1
$\mathcal{C}^c_{3,1}$	{m1, m4, m5} {pm, m3, m6, m7, a1, a3} {m2, a2}	2
$\mathcal{C}^c_{3,2}$	{m1, m4} {pm, m3, m5, m6, m7, a1, a3} {m2, a2}	2
$\mathcal{C}^c_{4,1}$	{m1, m2} {pm, m3, a3} {m4, m5, m6, m7, a1} {a2}	4
$\mathcal{C}^c_{4,2}$	{m1, a2} {pm, m3, a3} {m2, m4, m5} {m6, m7, a1}	4
$\mathcal{C}^c_{4,3}$	{m1, a2} {pm} {m2, m3, m4, m5, m7} {m6, a1, a3}	4
$\mathcal{C}^c_{5,1}$	{m1, a2} {pm} {m2, m4, m5, m7} {m3, a1} {m6, a3}	4

Table 9: Discussion Network matrix, rearranged according to $\mathcal{C}^a_{4,2}$

		m1	m2	a2	pm	m4	m3	m5	m6	m7	a1	a3
minister 1	1	0	1	0	1	0	0	1	0	0	0	0
minister 2	3	1	0	0	1	0	1	1	1	1	0	0
adviser 2	10	1	1	0	0	1	1	0	0	0	0	0
p.minister	2	0	0	0	0	0	0	0	0	1	0	0
minister 4	5	0	0	0	1	0	1	1	1	1	0	0
minister 3	4	0	0	0	0	0	0	0	1	1	0	0
minister 5	6	0	0	0	1	1	1	0	1	1	0	0
minister 6	7	0	0	0	0	0	1	0	0	1	1	1
minister 7	8	0	0	0	1	0	1	0	1	0	0	1
adviser 1	9	0	0	0	0	0	1	0	1	1	0	1
adviser 3	11	0	0	0	0	0	0	1	0	1	1	0

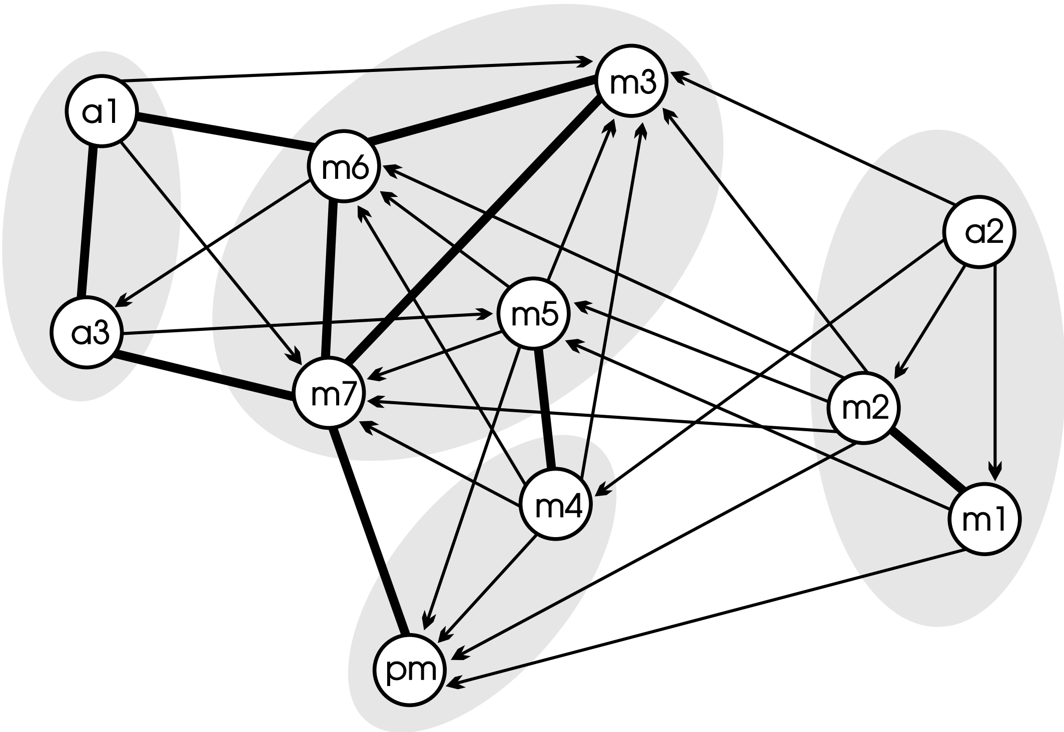

Figure 3: Partition

Table 10: Model and value matrix

	A	B	C	D	A	B	C	D
$A = \{\mathsf{m1}, \mathsf{m2}, \mathsf{a2}\}$	rdo	reg	rdo	–	1.33	1	1.5	0
$B = \{\mathsf{pm}, \mathsf{m4}\}$	–	rdo	rdo	–	0	0.5	1.25	0
$C = \{\mathsf{m3}, \mathsf{m5}, \mathsf{m6}, \mathsf{m7}\}$	–	rdo	rdo	rdo	0	1.5	2.25	1.5
$D = \{\mathsf{a1}, \mathsf{a3}\}$	–	–	cdo	com	0	0	2.5	1

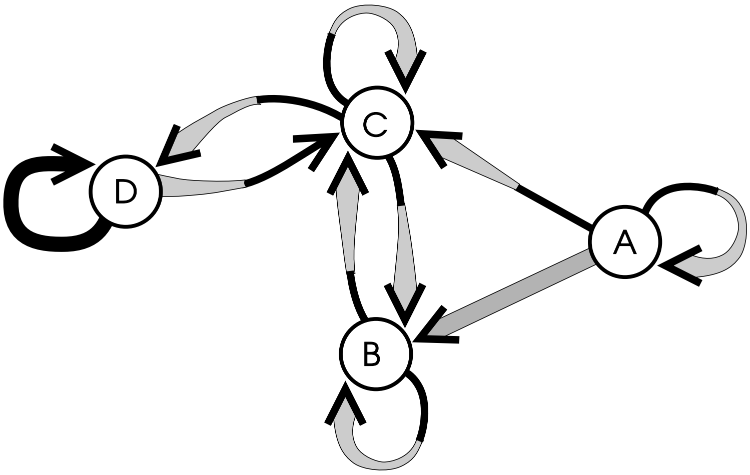

Figure 4: Model

All computations were carried out by program MODEL from a package of structure analysis programs STRAN (Batagelj 1991). MODEL2, the new version of MODEL, allows the user to specify the types of each connection in the model. The latest version of MODEL for PC is available in selfextracting format by anonymous FTP from

`ftp://vlado.mat.uni-lj.si/pub/networks/model.exe`

See also

`http://vlado.mat.uni-lj.si/ftpe.htm`

7 Conclusion

In this paper, we proposed a generalized approach to blockmodeling of social networks. Many things have still to be elaborated:

- other types of connection and criterion functions.
- which types of connection are compatible with the hierarchy – models of models; is there some 'algebra' behind it? It seems that Kim and Roush (1984) results may provide o good starting point.
- let $\mathcal{E}(\mathcal{N}, \mathcal{T})$ be the set of all equivalences compatible with $\mathcal{T}$ over $\mathcal{N}$. What can be said about the structure of this set? Can the results concerning the regular equivalences (Borgatti and Everett 1989) be extended to generalized equivalences?
- assigning values also to units.

References

Batagelj, V.
1991 "STRAN – STRucture ANalysis," Manual, Ljubljana.

Batagelj, V., P. Doreian, and A. Ferligoj
1992 "An optimizational approach to regular equivalence," *Social Networks*, 14:121–135.

Batagelj, V., A. Ferligoj, and P. Doreian
1992 "Direct and indirect methods for structural equivalence," *Social Networks*, 14:63–90.

Borgatti, S.P. and M.G. Everett
1989 "The class of all regular equivalences: Algebraic structure and computation," *Social Networks*, 11:65–88.

Doreian, P., V. Batagelj, and A. Ferligoj
1994 "Partitioning Networks on Generalized Concepts of Equivalence," *Journal of Mathematical Sociology*, 19/1:1–27.

Doreian, P. and A. Mrvar
1994 "A Partitioning Approach to Structural Balance," Manuscript.

Faust, K.
1988 "Comparison of methods for positional analysis: Structural and general equivalences," *Social Networks*, 10:313–341.

Hlebec, V.
1993 "Recall versus recognition: Comparison of two alternative procedures for collecting social network data," Developments in Statistics and Methodology. (A. Ferligoj and A. Kramberger, editors) Metodološki zvezki 9, Ljubljana: FDV, 121-128.

Kim, K.H. and F.W. Roush
1984 "Group Relationships and Homomorphisms of Boolean Matrix Semigroups," *Journal of Mathematical Psychology*, 28:448–452.

Lorrain, F. and H.C. White
1971 "Structural equivalence of individuals in social networks," *Journal of Mathematical Sociology*, 1:49–80.

Pattison, P.E.
1988 "Network models; some comments on papers in this special issue," *Social Networks*, 10:383–411.

White, D.R. and K.P. Reitz
1983 "Graph and semigroup homomorphisms on networks of relations," *Social Networks*, 5:193–234.

Wilson, R.J. and J.J. Watkins
1990 *Graphs, An Introductory Approach,* New York: John Wiley.