10 Ranking

10.1 Introduction

In the social sciences, society is regarded as a set of social layers or strata. Instead of ranking people, groups, or organizations on a continuous scale of prestige, they are usually classified into a limited set of discrete ranks, for instance, working class, lower middle class, upper middle class, and upper class. Within a group of humans, discrete ranking also occurs, e.g., leaders, followers, and outcasts. Probably, the stratification of art worlds into stars, settled artists, and mediocre artists is another example. In this chapter, we discuss techniques to extract discrete ranks from social relations.

Social ranking may be formal or informal and the two types of ranking may coexist. In a formal ranking, it is written down who commands whom and insignia or symbols minimize the ambiguity of the ranking and preclude any confusion about a person's rank. The army is an obvious example with its elaborate hierarchy. In contrast, an informal ranking is neither written down nor expressed by official symbols. It manifests itself in the opinions and behavior of people towards each other: respect and acts of deference versus disrespect and dominance.

The creation and maintenance of an informal ranking is a very important social process. Social network analysis is needed to investigate it and to assess the positions which individuals occupy within the informal ranking. If a formal ranking exists, it is interesting to compare it to the informal ranking because they do not need to match, just like informal communication patterns often deviate from the official communication structure.

The structural concept of social ranking is an extension to balance theory which we presented in Chapter 4. Balance theory assigns people to clusters which are not ranked with respect to one another. Within a cluster, people tend to like each other but people do not like members of other clusters. Within clusters as well as between clusters, relations are supposed to be symmetric: you are supposed to reciprocate the sentiment or choices which you receive. Elaborating on this perspective, asymmetric relations, for instance, A reports to B but B does not report to A, indicate ranking: B is ranked over A.

10.2 Example

Our example is a network of discussion relations among the eleven students who were members of the student government at the University of Ljubljana in Slovenia (`Student_government.net`). The students were asked to indicate with whom of their fellows they discussed matters concerning the administration of the university informally. We suppose that this relation indicates esteem: students will choose fellows whom they respect. Therefore, we expect this network to display informal ranking.

Within the parliament, students have positions which convey formal ranking: the prime minister, the ministers, and the advisors. In Figure 1, vertex color indicates the formal position of a student in the parliament (partition `student_government.clu`). We will compare the formal ranking to the informal ranking which we derive from network analysis of the discussion relations.

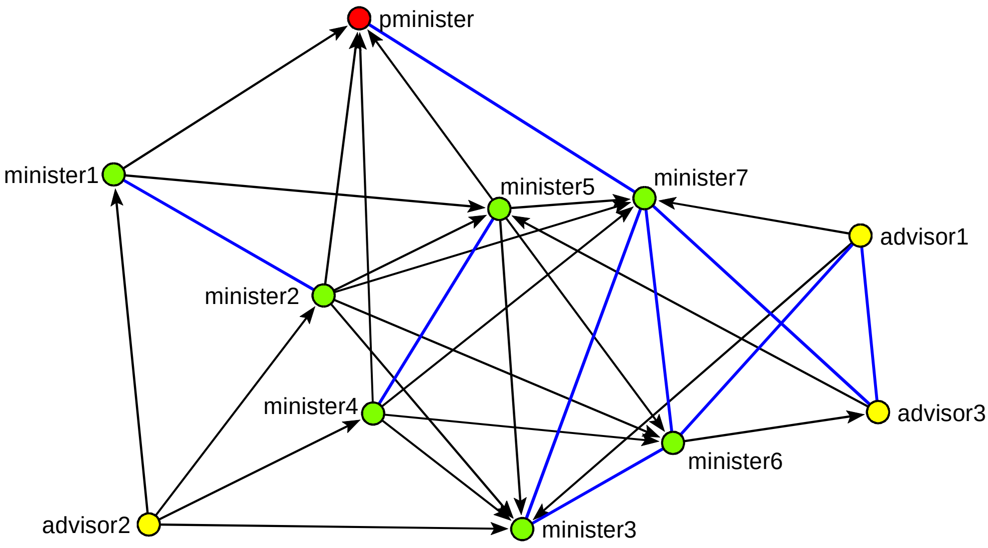


Figure 1 - Student government discussion network.

10.3 Triadic analysis

Before we can analyze the ranks in the student government discussion network, we must discuss balance theory once more. In Chapter 4, we learned that a balanced or clusterable network can be partitioned into clusters such that all positive choices occur within clusters and all negative choices are found between clusters. If we replace negative choices by absent choices, it follows that positive choices are found within clusters but choices do not occur between clusters. Since absent choices should not occur within a group, each positive choice must be reciprocated.

As a consequence, we can rephrase balance theory for the type of tie between two vertices (dyads) in a simple directed network: mutual choices indicate group membership and mutual absent or null choices indicate membership of different groups. Of course, this presupposes that the social relation under investigation implies a positive choice.

A **dyad** is a pair of vertices and the lines between them.

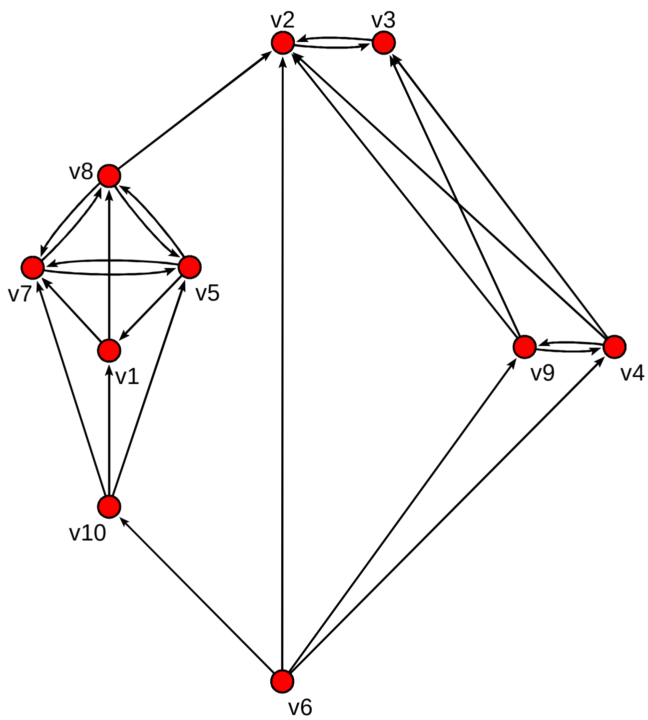


Figure 2 - An example of a network with ranks.

In the directed network of Figure 2, vertices v5, v7, and v8 constitute a cluster because they are connected by mutual choices (**complete dyads**) and vertices v4 and v9 constitute another cluster. These clusters are separated by absent lines or **null dyads**.

Both mutual choices and mutual absent choices are symmetric: you give as good as you get. Symmetric dyads indicate equivalence so we assume that vertices which are linked by symmetric ties belong to the same rank. The third type of dyad, however, is the **asymmetric dyad**: one person chooses the other but this choice is not reciprocated. Asymmetric dyads indicate ranking. In an asymmetric dyad, it is assumed that the receiver of the positive choice is ranked over the sender provided that being chosen expresses esteem or appreciation: the former can afford not to reciprocate the choice of the latter. In Figure 2, vertex v6 is ranked under v9 and v4 among others, which are ranked under v2 and v3.

In order to capture the structure of a directed network, we must proceed from dyads to **triads**. In a simple directed network, sixteen types of triads may occur, which are listed in Figure 3. A triad type is identified by a M-A-N number of three digits and, occasionally, a letter. The first digit indicates the number of mutual positive dyads (M), the second digit is the number of asymmetric dyads (A), and the third digit is the number of null dyads (N). Sometimes, a letter which refers to the direction of the asymmetric choices is added to distinguish between triads with the same M-A-N digits: D for down, U for Up, C for cyclic, and T for transitive (which we will explain later).

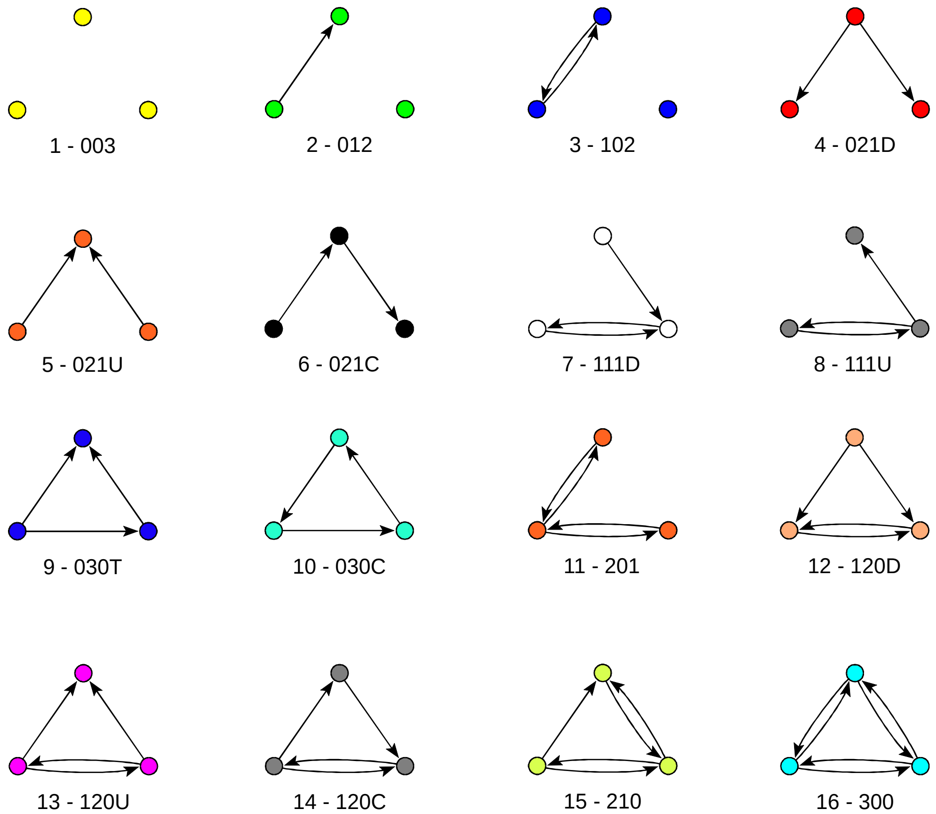


Figure 3 - Triad types with their sequential numbers in Pajek.

It has been shown, that the overall structure of a directed network can be inferred from the types of triads which occur. It is very important to understand the consequences of this discovery: it suffices to analyze small subnetworks (of size three) in order to understand the structure of the overall network!

If a directed network is balanced, for example, only two of the sixteen types of triads occur, namely, triads 300 and 102. Each cluster is a clique, so each subset of three vertices from a cluster is a complete triad like triad 300, e.g., vertices v5, v7, and v8 in Figure 2. If two vertices belong to one cluster and the third belongs to the other cluster, we encounter a triad in which two vertices are symmetrically linked because they belong to the same clique, e.g., vertices v5 and v8 in Figure 2, but they are not connected to the third vertex, which belongs to the other clique, e.g, vertex v9 in Figure 2. This is represented by triad 102. There are no other possibilities, so the two triad types identify the **balance model** for the structure of the entire network. If a network contains just these two types of triads then we now that the network consists of two cliques which are not interrelated.

In the course of time, four additional models for the overall structure of a directed network have been discovered, which we will present now. These models have a very important property, viz., that they progressively allow for more types of triads to occur. In other words, each model that we will present is less restrictive than the previous one. The second model, the **model of clusterability**, for instance, relaxes the demand of the balance model that the network consists of no more than two cliques. A clusterable network may contain three or more clusters. As a consequence, triad 003 is allowed to occur in a clusterable network because it contains vertices which belong to three different clusters. The two balanced triads (300 and 102) are also permitted because they still refer to vertices within one cluster of vertices of two clusters. The clusterability model is more permissive than the balance model: it allows for one more triad type.

In a similar way, the **model of ranked clusters** extends the clusterability model because it allows clusters to be spread over different ranks. Clusters at different ranks are connected by asymmetric dyads: each vertex in a lower cluster sends unilateral choices towards all vertices in a higher cluster. As a consequence, five triads are permitted which contain asymmetric dyads: 120D, 120U, 021D, 021U, and 030T. In triad 120U, for instance, the bottom two vertices belong to one cluster because they are linked by mutual choices. The top vertex is connected to them by asymmetric ties: it is chosen but it does not reciprocate the choices, so it must belong to a higher rank.

If a network contains these triads in addition to balanced or clusterable triads, the network can be partitioned into ranks and clusters according to the criteria that mutual choices are found within clusters, asymmetric choices point up to a higher rank, and null choices occur between clusters within a rank. In our example of a ranked network (Figure 2), vertices v4, v6, and v9 constitute a 120D triad and a 030T triad contains vertices v1, v5, and v10.

There are two models which relax the criteria of the ranked clusters model. The first model is the **transitivity model**. In a transitive triad, each path of length two is 'closed' by an arc from the starting vertex to the end vertex of the path. If actor A obeys actor B, and actor B obeys actor C, actor A also obeys actor C in a transitive triad. In a hierarchy, relations are usually transitive but transitivity is an effect which is also found in many other social relations, for instance, the relation 'to know someone' is often transitive.

The balanced, clusterable, and ranked clusters triads are transitive, but the 012 triad, which contains a single asymmetric choice, is also transitive because it does not offend the rule that an indirect choice is paralleled by a direct choice; the 012 triad simply does not contain an indirect choice. Under the ranked clusters model, the 012 triad would mean that the three vertices belong to different clusters within a rank because of the two null dyads and, at the same time, that two vertices are ranked as a result of the asymmetric dyad. Clearly, this is a contradiction. Under the transitivity model, however, null choices are allowed between ranks. It is not necessary that someone at a lower rank chooses all people of higher rank, for instance, boys and girls may have separate rank systems which are perfect ranked clusters but boys may ignore girls and vice versa.

The other model which relaxes the criteria of the ranked clusters model is the **hierarchical clusters model**, which is also called the hierarchical M-clusters model. This model permits asymmetric dyads within a group as long as they are acyclic. Within a cluster, asymmetric dyads are supposed to express a mild form of ranking within a group and, like any kind of ranking, this ranking must be acyclic. The set of vertices v1, v5, v7, and v8 (Figure 2) is an example of a hierarchical cluster. Vertex v1 is connected to the other vertices by two 120C triads. Vertex v7 does not reciprocated the choice by v1, who does not reciprocate v5's choice, so these three vertices are ranked but vertices v5 and v7 are also part of a cluster with vertex v8 because of the symmetric dyads.

The remaining five types of triads do not occur under any of the models. We may say that they are 'forbidden': they contradict all balance-theoretic models and the assumptions about symmetric and asymmetric dyads on which the models are based. If these triads occur often, we ought to doubt whether we can cluster and rank the data according to balance-theoretic principles. Table 1 summarizes the models.

Table 1 - Balance-theoretic models.

Model	Ties within a cluster	Ties between ranks	Permitted triads
Balance	symmetric ties within a cluster, no ties between clusters max. two clusters	none	102, 300
Clusterability	idem no restriction on the number of clusters	idem	+ 003
Ranked Clusters	idem	asymmetric ties from each vertex to all vertices on higher ranks	+ 021D, 021U, 030T, 120D, 120U
Transitivity	idem	null ties may occur between ranks	+ 012
Hierarchical Clusters	asymmetric ties within a cluster allowed provided that they are acyclic	idem	+ 120C, 210
no balance-theoretic model ('forbidden')			021C, 111D, 111U, 030C, 201

+ indicates that all triads in previous rows are also permitted.

Let us apply the balance-theoretic models to the example network (Figure 2). Table 2 shows the number of triads found in this network arranged by the balance theoretic model to which they belong. Such a distribution is known as the **triad census**. The models are less restrictive in the order in which they are listed in Table 2 and it is standard practice to characterize the overall structure of a network by the least restrictive model which applies. After all, a less restrictive model covers all more restrictive models because it also permits their triads.

Unfortunately, social networks hardly ever conform perfectly to a balance-theoretic model. Each triad type is likely to occur at least once, so the presence of one triad does not mean that the associated model must apply. We must compare the triad census to the distribution of triad types which is expected by chance. If a particular triad type occurs clearly more often than expected by chance, the corresponding model may be said to guide or influence the relations: there is a tendency towards balance, clusterability, ranked clustering, transitivity, or hierarchical clusters in the network. If the models explain network structure, the 'forbidden' triads should occur less frequently than predicted by chance.

Table 2 - Triad census of the example network.

	Type	Number of triads	Expected	Model
3	102	**22**	7.56	Balance
16	300	1	0.06	
1	003	**7**	17.03	Clusterability
4	021D	**3**	7.56	Ranked
5	021U	**3**	7.56	Clusters
9	030T	4	5.81	
12	120D	**5**	1.12	
13	120U	2	1.12	
2	012	**58**	39.3	Transitivity
14	120C	2	2.24	Hierarchical
15	210	0	0.86	Clusters
6	021C	**7**	15.12	Forbidden
7	111D	4	5.81	
8	111U	**2**	5.81	
10	030C	0	1.94	
11	201	0	1.12	
Total		120		

Table 2 shows the triad census of the example network. The column headed 'Number of triads' shows the triad counts in the example network and the column 'Expected' lists the numbers of triads which are expected by chance in a network of this size containing this number of arcs. If the actual frequencies are close to the expected frequencies, the network does not conform to any of the balance-theoretic models and we may conclude that its structure is random from the point of view of balance theory. This, however, does not seem to be the case in our example.

In the example network, some types of triads occur substantially more often or less frequently than expected by chance. These frequencies are printed in bold face in the table. The example network seems to contain relatively few clusterable triads but many balanced ones, some ranking (120D) although other ranked clusters triads occur less often than expected (021D and 021U), and a tendency towards transitivity but not a surprising number of hierarchical cluster triads. The forbidden triads occur at chance level or less (021C and 111U), so we do not have to discard all balance-theoretic models. The most appropriate model for this network seems to be the transitivity model, which allows for clustering and ranking but which does not require that all ties between ranks are asymmetric.

We should note that our expected frequencies only take into account the number of vertices and arcs in the network. Standard statistical tests of the triad census condition on indegree, outdegree, and number of mutual choices, which expresses the tendency to reciprocate choices at the level of the dyad. These statistical tests may produce different results but they fall outside the scope of the present book.

The triad census is an example of a research strategy which concentrates on local structure since it accounts only for relations within triads. The implications for the overall structure of the network are usually taken for granted and not much effort is made to assign vertices to clusters and ranks. Triadic analysis is the basis of statistical models which test hypotheses about the relations of individual actors: why do they establish some relations and not others? Are their choices motivated by balance, transitivity?

Application

Info>Network>Triadic Census

In Pajek, it is very easy to compute the triad census: simply use the *Triadic Census* command in the *Info>Network* submenu. A dialog box asks whether the models should be reported and if you choose this option, the triad types ('Type'), their actual frequencies ('Number of triads (ni)'), and the frequencies expected by chance ('Expected (ei)') are reported. In addition, the relative difference between the actual and the expected number of triads is shown ('(ni-ei)/ei') and a statistic testing the hypothesis that the actual frequencies are equal to the expected frequencies. This statistic is not reliable if expected frequencies are low.

Table 3 contains the triad census for the student government network. Three of the five forbidden triads appear less frequently than expected by chance in the student government network (triads 021C, 111U, and 030C), which is also signaled by the negative value of the actual versus expected ratio, so there is some support that the underlying ideas of symmetric and asymmetric ties apply here.

Table 3 - Triad census of the student government network.

	Type	Number of triads (ni)	Expected (ei)	(ni-ei)/ei	Model
3	102	20	10.65	0.88	Balance
16	300	1	0.44	1.26	
1	003	10	10.05	-0.01	Clusterability
4	021D	9	10.65	-0.15	Ranked
5	021U	15	10.65	0.41	Clusters
9	030T	7	12.65	-0.45	
12	120D	14	3.76	2.72	
13	120U	6	3.76	0.60	
2	012	37	35.84	0.03	Transitivity
14	120C	1	7.52	-0.87	Hierarchical
15	210	5	4.47	0.12	Clusters
6	021C	16	21.29	-0.25	Forbidden
7	111D	13	12.65	0.03	
8	111U	6	12.65	-0.53	
10	030C	1	4.22	-0.76	
11	201	4	3.76	0.06	

Chi-Square: 55.7613***
6 cells (37.50%) have expected frequencies less than 5.
The minimum expected cell frequency is 0.44.

Then, which structure characterizes the network? The student government network contains more between groups triads (triad 102) than expected by chance but the number of clusterability triads (003) is predicted by chance, so a partition into two clusters seems to suffice. Some ranked clusters triads appear as often as expected by chance but the number of 120D triads, which signal asymmetric choices towards mutually connected pairs, is much higher than the expected frequency, so we should conclude that the network is ranked. Finally, the number of triads identifying the transitivity model (012) matches the amount expected in a random network and the hierarchical cluster triads also do not appear more often than expected by chance.

A ranked clusters model seems to be the best choice for this data set because it permits triads 120D, 120U, and 021U, which appear substantially more often than chance, but it also permits the triads associated to more restrictive models, viz., the two balanced triads 300 and 102. In this way, the ranked clusters model contains all types of triads which occur clearly more often than expected by chance in the student government network.

10.4 Acyclic networks

In directed networks, ranking is associated with asymmetry: arcs which represent an 'ego obeys alter' relation point up, not down. Triadic analysis applies this principle to triads, that is, to local structure, but it can also be applied to the overall structure of a directed network. In a network which reflects a hierarchy perfectly, all arcs should point up and no arc should point down from a higher rank to a lower rank. This is called an acyclic network. It is important to note that such a network cannot contain cycles because a cycle would include arcs pointing up *and* arcs pointing down in order to return to its starting point.

An **acyclic network** does not contain cycles.

We associate ranking with acyclic structures, e.g., the soldier is subordinate to the sergeant, who is subordinate to the captain, who is subordinate to the colonel, etc. An arc pointing in the wrong direction, e.g., the colonel obeys the soldier for whatever mysterious reasons, contradicts our idea of a hierarchy. This arc creates a cycle in the network and it may even make the whole network cyclic in the sense that in the end everyone obeys everybody.

When acyclic structures point to ranking, cyclic structures are associated with clusters within one rank because they suggest equality among its vertices. In the short run, e.g., in a symmetric dyad, or in the long run, e.g., in a 'feedback loop' which includes many vertices, a choice is reciprocated. From this point of view, we may partition a directed network into ranks: cyclic subnetworks represent a cluster within a rank and acyclic structures link ranks into a hierarchy.

Fortunately, it is easy to detect the cyclic parts of a network and you already master the technique to do it. Recall that a strong component is a maximal subnetwork in which each vertex is reachable to each other vertex (Chapter 3). There are paths in both directions between all pairs of vertices within a strong component so a strong component is a cyclic (sub)network by definition. The arcs which are not part of a strong component cannot belong to a cycle, so they are part of an acyclic structure. In fact, if we shrink the strong components of a network, the network becomes acyclic.

Figure 4 shows the strong components in the student government discussion network. There are three components and, if you look carefully, the arcs between strong components all point in the same direction: from the red component to the green component to the yellow component. This is very clear in Figure 5, which shows the network with shrunk strong components. Note that the prime minister is included in the strong component in the top of the hierarchy, which is in line with his formal position.

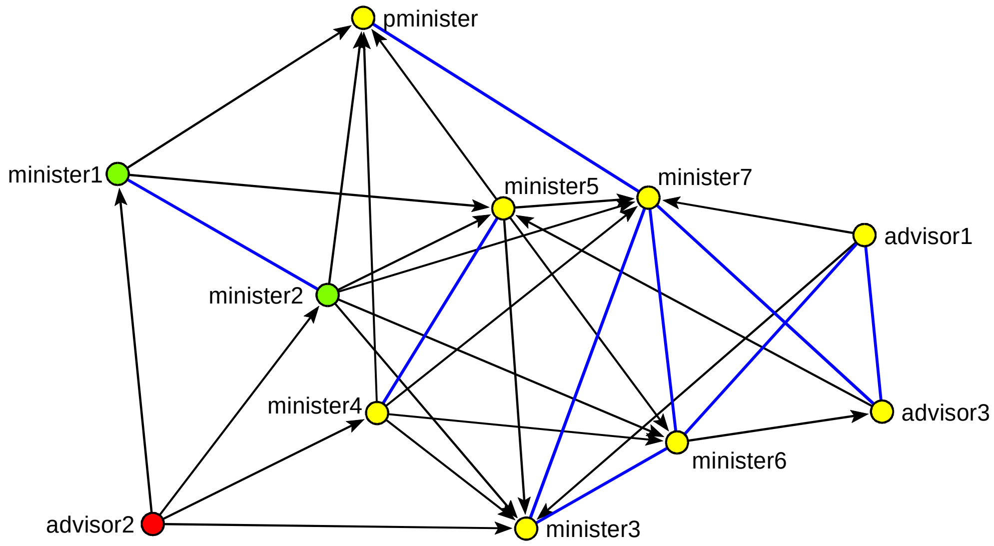


Figure 4 - Strong components in the student government discussion network.

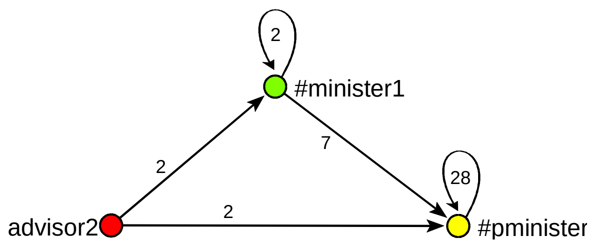


Figure 5 - Acyclic network with shrunk components.

Application

Net>Components>Strong

In Chapter 3, we learned to identify the strong components in a network with the command *Net>Components>Strong*. This command creates a partition with a class for each strong component. We advise to set the minimum size of a component to one, otherwise the 'red' component of advisor2 is not recognized.

Operations>Shrink >Partition

Chapter 2 presented the command to shrink a network. The present case does not offer any complications. If the network is shrunk according to a strong components partition, we obtain three vertices as shown in Figure 5. We colored the vertices manually to simplify the comparison between the shrunk network and the original network.

10.5 Symmetric-acyclic decomposition

In triadic analysis (Section 10.3), clusters within a rank must be complete. In many social networks, this criterion is too strict. Usually, social networks contain a limited number of choices made by each individual as a result of measurement techniques and social or cognitive limitations at the part of the investigated people. Respondents who are asked to recall with whom they discussed a particular matter informally, for instance, are likely to mention their most salient contacts rather than everyone with whom they have merely touched on the subject. A network constructed from these data will not yield complete clusters.

On the other hand, however, the strong components do not seem to be sufficiently strict to identify a cluster within a rank (Section 10.4). In the student government discussion network, it would be nice if we could subdivide the 'yellow' component, which contains a heterogeneous group of actors at present: advisors, ministers, and the prime minister.

The symmetric-acyclic model is a suitable alternative. It uses a version of the symmetry versus asymmetry principle which is less strict than the balance-theoretic assumptions but stricter than the acyclic character of strong components. It assumes that vertices which are linked by symmetric (i.e., mutual) choices directly or indirectly belong to one cluster, hence to one rank. Clusters which are linked by asymmetric ties only, are ranked.

This model is especially less restrictive with respect to the internal structure of clusters because it allows for asymmetric and null dyads within a cluster, for example, if vertex *u* is linked to vertices *v* and *w* by symmetric ties, they belong to one cluster regardless of the tie between *v* and *w*, which may be symmetric, asymmetric, or null. Balance-theoretic models never allow null dyads within a cluster and asymmetric dyads may occur only under special conditions in the hierarchical clusters model.

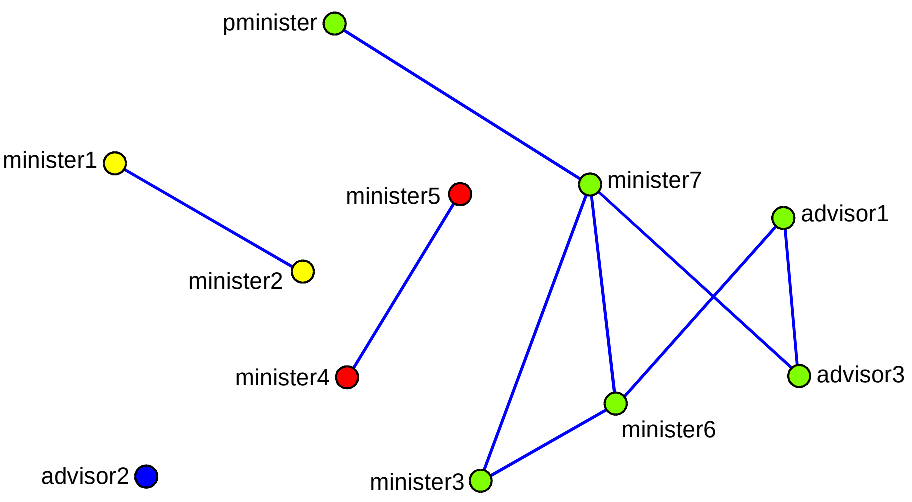


Figure 6 - Clusters of symmetric ties in the student government network.

It is easy to identify clusters of vertices which are connected by mutual choice: just delete all unilateral arcs from the network and compute components. Each component is a cluster of vertices which are linked by symmetric dyads. Figure 6 shows the four clusters in the student government network. Note that the blue and yellow symmetric clusters of Figure 6 are strong components in the overall network (Figure 4). It is clear that advisor2 is ranked under ministers one and two because they are linked by one-way relations.

The largest component, however, combines two symmetric clusters: the cluster of ministers four and five with the cluster of the prime minister. The two symmetric clusters are linked into one strong component because the arcs between these clusters do not point in the same direction. In Figure 7, we can see that the symmetric cluster of ministers four and five predominantly sends asymmetric ties to the cluster of the prime minister but they receive one asymmetric choice from that cluster. When we ignore this arc, we obtain strong components which match the symmetric clusters perfectly.

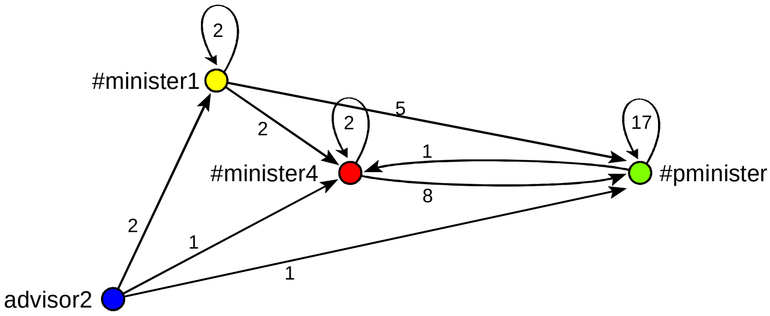


Figure 7 - Discussion network shrunk according to symmetric clusters.

Clusters of vertices which are reachable through symmetric ties are preferable over strong components because mutual choice is a clear indication of group membership and equality with respect to ranking. We therefore recommend to pay close attention to strong components in which not all vertices are linked by paths of mutual choices. Elimination of a single arc may split this component into smaller clusters which are asymmetrically ordered.

A stricter interpretation of the symmetric-acyclic model forbids all asymmetric ties inside clusters. In other words, vertices within a cluster are either tied by a symmetric relation or no relation at all (a null relation) and all asymmetric choices are situated between clusters. In the largest cluster of the student government network, which contains the prime minister, three asymmetric ties are found and all of them involve the advisors. The ministers and the prime minister are linked by symmetric ties only. If we would delete advisor3, who is offending the ranking between two symmetric clusters and who is also involved in an asymmetric tie within the top cluster, and we ignore the arc from minister6 to advisor1, we obtain a decomposition which satisfies the strictest criteria of the symmetric-acyclic model (see Figure 8).

Note that this decomposition nicely reflects the formal positions of the students: the advisors are on the lower ranks, the prime minister is on the highest rank, and the ministers are in the middle or top ranks. If the prime minister had not chosen minister7, he would have had the top rank for himself or herself. In this case, the informal ranking is more differentiated than the formal ranking since the ministers are spread over three ranks.

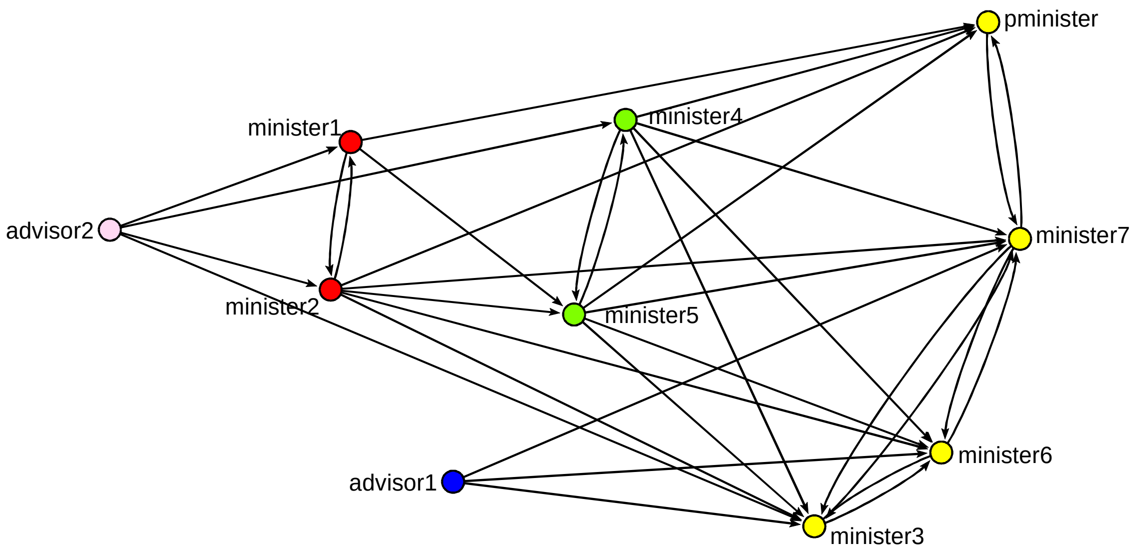


Figure 8 - Symmetric components in the (modified) student government discussion network.

Clusters which are not ranked with respect to one another because there is no path of asymmetric choices between them may belong to the same rank. In Figure 8, for instance, the blue cluster, which contains advisor1, can belong to the same rank as the green cluster. There are only null relations between these clusters, which satisfies the balance-theoretic principle. The blue cluster may, however, also be placed at the ranks of the red and pink clusters or in between any pair of ranks as long as it is beneath the rank of the yellow cluster. The classification of vertices according to rank does not necessarily yield a single result.

Application

Hierarchical Decomposition >Symmetric-Acyclic

Pajek contains a command to find clusters of symmetrically linked vertices and ranks: *Hierarchical Decomposition>Symmetric-Acyclic*. This command follows the logic outlined above.

First, the command finds the components of symmetrically linked vertices. It produces a new network with edges instead of bi-directed arcs. Then it creates a network without the remaining (unilateral) arcs and it computes a partition of weak components. Each weak component is a cluster of vertices which are reachable through symmetric ties. When you draw the network and partition, you obtain Figure 6.

Second, the procedure shrinks the clusters of symmetrically linked clusters. The shrunk network is very convenient for finding symmetric clusters which are linked by asymmetric ties in both directions. If you draw this network, you obtain a sociogram which is similar to Figure 9. In this drawing, you may detect symmetric clusters which are nearly asymmetrically linked, such as the #minister4 and #pminister clusters.

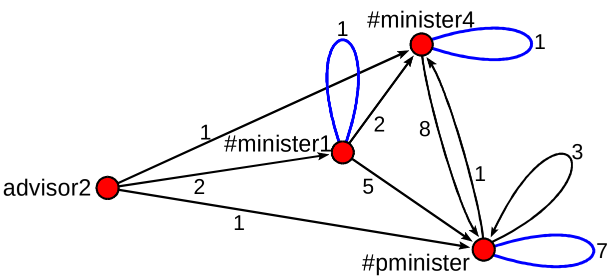


Figure 9 - First clustering in the symmetric-acyclic decomposition.

Finally, the procedure repeats the first and second steps until it does not encounter any symmetrically linked vertices or clusters. Then, all strong components have been shrunk and the network is acyclic by definition. Note that the last network created by the *Symmetric-Acyclic* command does not contain any lines sometimes. In this case, you should select the last shrunk network as the final result of the analysis.

Net>Transform>Remove >loops

Net>Partitions>Depth >Acyclic

The shrunk network which results from the symmetric-acyclic decomposition is acyclic, so we may determine the order of the ranks with the *Depth>Acyclic* command from the *Net>Partitions* submenu provided that you delete the loops first (*Net>Transform>Remove>loops*), which are created when the network is being shrunk. Draw the shrunk network and its depth partition according to layers to obtain a graphical representation of the ranks.

Net>Components>Strong

If you want to draw the original network in layers which represent the ranks, you have to expand the depth partition to the original network. Since the shrunk vertices in the acyclic network are the strong components (size one and larger) in the original network, you can use a strong components partition of the original network to expand the depth partition. Create this partition with the *Net>Components>Strong* command making sure that the original network is selected in the Network drop list.

Partitions>Expand >First according to Second (Shrink)

Now, you can expand the depth partition of the shrunk network to the original network. Select the depth partition of the shrunk network as the first partition in the *Partitions* menu and select the strong components partition as the second partition. Then choose the command *First according to Second (Shrink)* from the *Partitions>Expand* submenu. Pajek asks which class in the strong components partition was not shrunk (zero or a number which does not occur in the strong components partition will do) and it creates a new partition which assigns each vertex in the original network to its depth in the symmetric-acyclic decomposition. You may draw this partition in layers and move vertices within each layer to obtain an image of the ranking. Note that all arcs between ranks point in the same direction.

In the symmetric-acyclic decomposition, the resulting strong components are not necessarily symmetric clusters. In the student government network, for example, a strong component combines the two symmetric clusters #pminister and #minister4 (Figure 9). We have found a decomposition which satisfies the weak version of the symmetric-acyclic model.

Operations>Transform >Remove Lines >Between Clusters

The stronger version of this model does not allow asymmetric ties within clusters, so we have to inspect the ties within each cluster in order to find out whether the stronger model applies. Since the strong components are the clusters, we may simply remove all lines between strong components and check whether the resulting network contains bi-directed arcs only. Select the original network and the strong components partition to this network and remove the lines between components with the *Operations>Transform >Remove Lines >Between Clusters* command. Now, replace bi-directed arcs by edges (command *Net>Transform>Arcs->Edges>Bidirected only*) and check the number of arcs in the network (*Info>Network>General*). If there are no arcs, all strong components are symmetric clusters. Strong components containing arcs, however, are not symmetric clusters, so they do not satisfy the stronger version of the symmetric-acyclic model. Maybe you can find an arc which must be removed in order to obtain a symmetric cluster, e.g., the arc from advisor3 to minister5 in the student government network.

10.6 Summary

Society and, in more detail, the human group is characterized by clustering and ranking. Like-minded people cluster into cohesive groups on the basis of mutual positive relations. Rivalry between groups is expressed by negative or absent relations. In addition, social groups are usually ranked such that dominant groups occupy higher ranks or strata. Asymmetric ties indicate ranking: a positive choice received from a lower ranked group is not reciprocated.

Society and the social group are generally considered to contain a limited number of discrete ranks. In this chapter, we present structural models of discrete ranks which have evolved from balance theory. The first two balance-theoretic models – balance and clusterability (see Chapter 4) – are confined to the clustering of social entities; they tacitly assume that there is no ranking, so asymmetric ties and unclusterable semicycles are not allowed. A third model, the ranked clusters model, regards a social system as a set of ranks where each rank contains one or more clusters. Positive arcs connect entities within a cluster but no arcs connect different clusters at one level, as in the clusterability model. In addition, asymmetric dyads connect clusters at different ranks, where arcs point from lower to higher levels.

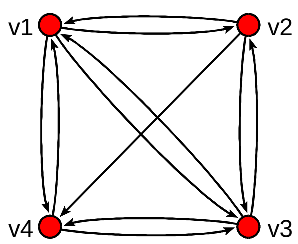
The ranked clusters model represents a simple hierarchy in which each pair of clusters or vertices is unambiguously ranked. Often, social systems are more complicated containing incomplete hierarchies or even different hierarchies which are not compatible. The social cleavage between girls and boys is a simple example. There is a hierarchy of boys and a hierarchy of girls but nobody is interested in the members of the other gender regardless of their ranking. This phenomenon is captured in the fourth balance-theoretic model, which is known as the transitivity model. A fifth model, called the hierarchical clusters model, is even more permissive because it allows for ranking within a group. Asymmetric dyads within a cluster of otherwise symmetrically connected people indicate ranking in this model.

In a simple directed network, a balance-theoretic model is identified by the types of triads which it permits, so we may count the number of times each triad type occurs in the network - this is called the triad census - and find the appropriate model. Unfortunately, social networks seldom fit a balance-theoretic model perfectly, so we need statistical tests to determine which triad types and models occur more often than expected by chance. Triadic analysis is the basis for statistical modeling rather than exploring the structure of clusters and ranks.

By definition, ranking is acyclic, so cyclic parts of the network either represent clustering within a rank or they contain complicated or imperfect ranking. Recall that a strong component contains vertices which are connected by paths in both directions, so strong components are cyclic subnetworks. If we shrink the strong components, the resulting network is acyclic and can be partitioned into ranks. Next, we inspect each strong component for clusters and complicated or imperfect ranking. In a simple directed network, mutual (positive) choices are the backbones of clusters, so we look for clusters of vertices which are directly or indirectly linked by symmetric ties. The relations between the clusters tell us whether they belong to one rank or to different ranks.

This is an exploratory procedure for detecting the clusters and ranks which best fit a network but it does not tell us whether the fit is satisfactory. With enough effort and modifications we can probably find clusters and ranks which even fit a random network. As elsewhere in this book, we must make sense of our results. The clusters and ranks should be meaningful with respect to other information that we have about the social entities in the network.

10.7 Exercises



1 How many dyads does the network shown at the right contain and how many types of dyads?
   - a six dyads and one type
   - b six dyads and two types
   - c eleven dyads and one type
   - d eleven dyads and two types

2 Assemble the triad census (type of triad and frequency of occurrence) of the network shown above by hand.

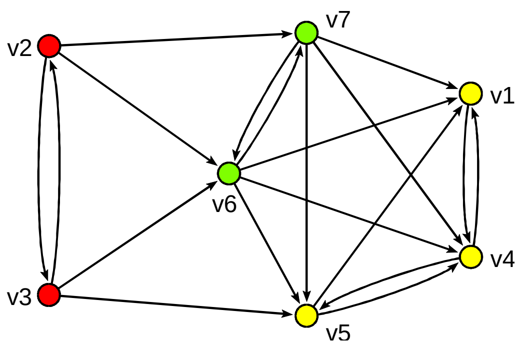
3 Which balance-theoretic model characterizes the network of Exercise 1?
   - a the balance model
   - b the hierarchical clusters model
   - c the balance and hierarchical clusters models
   - d no balance-theoretic model fits this network

4 The table below shows the triad census of a directed network. Choose the appropriate balance-theoretic model for this network and justify your choice.

No.	Type	Number of triads (ni)	Expected (ei)	(ni-ei)/ei	Model
3	102	41	10.7	2.83	Balance
16	300	2	0.1	15.63	
1	003	8	19.4	-0.59	Clusterability
4	021D	0	10.7	-1.00	Ranked
5	021U	3	10.7	-0.72	Clusters
9	030T	7	9.2	-0.24	
12	120D	14	2.0	6.13	
13	120U	6	2.0	2.05	
2	012	77	49.9	0.54	Transitivity
14	120C	0	3.9	-1.00	Hierarchical
15	210	0	1.7	-1.00	Clusters
6	021C	2	21.4	-0.91	Forbidden
7	111D	3	9.2	-0.67	
8	111U	2	9.2	-0.78	
10	030C	0	3.1	-1.00	
11	201	0	2.0	-1.00	

5 Explain why triad 201 is not allowed under the hierarchical clusters model.

6 Assign the vertices of the network depicted below to clusters and ranks. vertex colors indicate strong components. Does your decomposition satisfy the weak or the strong symmetric-acyclic model?



7 Which of the following statements about symmetric-acyclic decomposition is correct?

a in the strong symmetric-acyclic model, clusters do not contain asymmetric dyads

b in the strong symmetric-acyclic model, vertices in different ranks are always connected by asymmetric dyads

c in the weak symmetric-acyclic model, clusters contain mutual or null dyads only

d in the weak symmetric-acyclic model, all asymmetric dyads point from a lower rank to a higher rank

10.8 Assignment

In 1976, a literary critic published an essay about contemporary Dutch prose. In his essay, he distinguished between four trends or movements: 'narrators' including the authors Donkers, Kooiman, Matsier, and Meijsing, 'alienators' including Van Marissing, Robberechts, and Vogelaar, 'petty realism' (Hart, Hiddema, Luijters, Meinkema, Plomp, and Sijtsma), and 'decadence' (Siebelink, Joyce & Co). Find out whether this classification matches the ranks and clusters in the networks of critical attention in 1976. The simple directed network `literature_1976.net` contains an arc between two people if the first has paid attention to the second in an interview or review. Hint: create a partition reflecting the classification of the authors according to literary movement.

10.9 Further reading

- Chapter 6 of S. Wasserman and K. Faust's *Social Network Analysis: Methods and Applications* (Cambridge: Cambridge University Press, 1994) provides an excellent overview over balance-theoretic models. For more information on the hierarchical M-clusters model, see E.C. Johnsen, 'Network macrostructure models for the Davis-Leinhardt set of empirical sociomatrices' (in *Social Networks,* 7 (1985), 203-224).
- For more information on the student government data, consult V. Hlebec, 'Recall versus recognition: comparison of two alternative procedures for collecting social network data.' (in A. Ferligoj & A. Kramberger (Eds.), *Developments in Statistics and Methodology*. Ljubljana: FDV, 1993). Results of an analysis of the Dutch literary criticism data are reported in W. de Nooy, 'A literary playground. Literary criticism and balance theory.' (in *Poetics*, 26 (1999), 385-404).

10.10Answers

1 Answer b is correct. A dyad is a pair of vertices and the lines among them. In a network with four vertices, such as the example, there are six different pairs of vertices, so there are six dyads. In a simple directed network, a dyad is either mutual (arcs in both directions), asymmetric (an arc in one direction), or null (no arcs). In the example, five dyads are mutual and the sixth (v2 and v4) is asymmetric, so there are two types of dyad.

2 The table below shows the triad census.

No.	Type	Number of triads	Model
3	102	0	Balance
16	300	2	
1	003	0	Clusterability
4	021D	0	Ranked
5	021U	0	Clusters
9	030T	0	
12	120D	0	
13	120U	0	
2	012	0	Transitivity
14	120C	0	Hierarchical
15	210	2	Clusters
6	021C	0	‘forbidden’
7	111D	0	
8	111U	0	
10	030C	0	
11	201	0	
Total		4	

3 Answer b is correct. In Exercise 2, you have found two balanced triads (300) and two hierarchical cluster triads (210). The hierarchical clusters model allows for balanced triads but the reverse is not true. Therefore, the hierarchical clusters model is the appropriate model for this network.

4 The transitivity model is appropriate here. The ‘forbidden’ triads do not occur (030C and 201) or occur less often than in random networks, so a balance-theoretic model characterizes this network. The hierarchical cluster triads do not occur, but the network contains far more transitivity triads (012) than expected by chance. Two ranked clusters triads (120D and 120U) and both balanced triads appear more often than expected by chance, but they are also permitted by the transitivity model, so we may conclude that the transitivity model characterizes this network.

5 Triad 201 contains two symmetric choices and one null dyad. In the hierarchical clusters model, vertices connected by symmetric ties belong to one (hierarchical) cluster. A null dyad means that two vertices belong to different clusters. Therefore, two vertices belong to different clusters because of the null dyad and to the same cluster because of the path of symmetric choices at the same time. This is a contradiction, so this triad is not allowed.

6 Arcs between strong components point from the red to the green component and from the green to the yellow component. Clearly, there are three ranks, the yellow rank is the top rank and the red rank is at the bottom. The vertices in the red and green component are connected by mutual arcs but two yellow vertices (v1 and v5) are connected by an asymmetric tie, so the decomposition does not satisfy the criteria of the strong symmetric-acyclic model for all strong components.

7 Statement a is correct. In the strong symmetric-acyclic model, clusters do not contain asymmetric dyads, hence all asymmetric dyads are found between ranks. This is not the case in the weak symmetric-acyclic model, where asymmetric dyads may occur within clusters (answers c and d). Answer b is not correct because vertices at different ranks can also be connected by null dyads.