N E T W O R K S E M I N A R

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Some Mathematics of Network Analysis

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1 Introduction

This paper contains an overview of the main results and ideas from the last three months work, together with Anuška Ferligoj and Patrick Doreian, on the project *Connecting Cluster and Network Analysis.*

1.1 Basic notions

Let $E = \{X_1, X_2, ..., X_n\}$ be a finite set of *units.* The units are related by binary *relations*

$$R_t \subseteq E \times E, \quad t = 1, ..., r$$

which determine a *network*

$$\mathcal{N} = (E, R_1, R_2, ..., R_r)$$

In the following we shall restrict our discussion to a single relation R. It is usually described by a corresponding binary matrix $\mathbf{R} = [r_{ij}]_{n \times n}$ where

$$r_{ij} = \begin{cases} 1 & X_i R X_j \\ 0 & \text{otherwise} \end{cases}$$

In some applications r_{ij} can be a nonnegative real number expressing the strength of the relation R between units X_i and X_j.

The main goal of block modelling is to identify, in a given network *clusters*, (classes) of units which play the same or similar role – have the same or similar connection patterns to other units. They form a *clustering*

$$\mathcal{C} = \{C_1, C_2, ...C_k\}$$

*In visit at the University of Pittsburgh, Department of Sociology (november and december 1990).

which is a partition of the set E:

$$\bigcup_i C_i = E$$

$$i \neq j \Rightarrow C_i \cap C_j = 0$$

Each partition determines an equivalence relation (and vice versa). Let us denote by $\sim$ the relation determined by partition $\mathcal{C}$. A *block model* consists of structures obtained by identifying all units from the same cluster of the clustering $\mathcal{C}$. The partition (ideally) is constructed by using structural information contained in R and actors in the same partitioned class are equivalent to each other in terms of R alone. Such actors share a common structural position within the network. (In general the statement can be extended to include $\{R_t\}$.)

There are two basic approaches to the equivalence of units in a given network:

- the equivalent units have the same or similar connection pattern to the **same** neighbours; they are interchangeable.
- the equivalent units have the same or similar connection pattern to (possibly) different neighbours; they play the same role in the network.

The first type of equivalence is formalized by the notion of structural equivalence; the second, by the notion of regular equivalence.

1.2 Structural equivalence

A permutation $\varphi : E \to E$ is an *automorphism* of the relation R iff

$$\forall X, Y \in E : (XRY \Rightarrow \varphi(X)R\varphi(Y))$$

The units X and Y are *structurally equivalent*, we write $X \equiv Y$, iff the permutation (transposition) $\pi = (XY)$ is an automorphism of the relation R.

With other words: X and Y are structurally equivalent iff

s1. $XRY \Leftrightarrow YRX$
s2. $XRX \Leftrightarrow YRY$
s3. $\forall Z \in E \setminus \{X, Y\} : (XRZ \Leftrightarrow YRZ)$
s4. $\forall Z \in E \setminus \{X, Y\} : (ZRX \Leftrightarrow ZRY)$

or in the matrix form: $X_i \equiv X_j$ iff

s1'. $r_{ij} = r_{ji}$
s2'. $r_{ii} = r_{jj}$
s3'. $\forall k \neq i, j : r_{ik} = r_{jk}$
s4'. $\forall k \neq i, j : r_{ki} = r_{kj}$

The matrix form of the definition of structural equivalence can be extended also to the case when r_{ij} are real numbers.

1.3 Regular equivalence

The equivalence relation $\cong$ on E is a *regular equivalence* on network $\mathcal{N} = (E, R)$ iff for all $X, Y, Z \in E$, $X \cong Y$ implies both

R1. $XRZ \Rightarrow \exists W \in E : (YRW \wedge W \cong Z)$
R2. $ZRX \Rightarrow \exists W \in E : (WRY \wedge W \cong Z)$

1.4 Establishing block models

The definitions of equivalences describe the ideal states. The real networks can be seen as perturbed ideal networks. For this reason the procedures for network block analysis usually seek for the (ideal) clustering $\mathcal{C}$ which fits the best the given network data.

There are two main approaches to block modeling problems based on the structural equivalence and its relaxations.

- *indirect approach*: reduction to the standard data analysis problems (cluster analysis, multidimensional scaling) by determining a dissimilarity matrix between units which is compatible with the selected type of equivalence;
- *direct approach*: construction of criterion function $P(\mathcal{C})$ which measures the fit of the clustering $\mathcal{C}$ to the network data, and solving the corresponding optimization problem. For this purpose a relocation procedure from cluster analysis can be adapted.

2 The indirect approach via measuring the equivalence of pairs of units joint work with A. Ferligoj and P. Doreian

This section contains results from [5].

Although the definition of structural equivalence is *local* it has *global* implications – structurally equivalent units behave in the same way also to all other units. A position is defined in terms of all other units in a network. This can be formally expressed in different ways.

2.1 Properties of pairs of units

Let $q(U, V)$ be some numeric structural property depending only on units U and V and the relation R, but not on the names (labels) of units. Formally expressed: A property $q : E \times E \rightarrow \mathbb{R}$ is *structural* if for every automorphism φ of the relation R and every pair of units $X, Y \in E$ it holds

$$q(X, Y) = q(\varphi(X), \varphi(Y))$$

For example

$$q(U, V) = r(U, V)$$

or as another example

$$q(U,V) = \text{ number of common neighbours of units } U \text{ and } V$$

or

$$q(U,V) = \text{ length of the shortest path from } U \text{ to } V$$

For other examples see [10].

Then, if $X \equiv Y$ we have

q1. $q(X,Y) = q(Y,X)$
q2. $q(X,X) = q(Y,Y)$
q3. $\forall Z \in E \setminus \{X,Y\} : q(X,Z) = q(Y,Z)$
q4. $\forall Z \in E \setminus \{X,Y\} : q(Z,X) = q(Z,Y)$

The property q is *sensitive* if the properties q1 – q4 imply also that $X \equiv Y$.

Therefore we can describe each unit U by a vector

$$[U] = [q(U,X_1), q(U,X_2), \ldots, q(U,X_n), q(X_1,U), \ldots, q(X_n,U)]$$

and define the dissimilarity between units $U, V \in E$ as

$$d(U,V) = D([U],[V])$$

where D is a dissimilarity between the corresponding descriptions. Some examples of such dissimilarities and the discussion of their appropriateness is given in the section 2.3.1.

2.2 Properties of units

Let $t(U)$ be the structural property of the unit U depending only on the unit U and relation R, but not on the names of units. The property $t : E \to \mathbb{R}$ is *structural* if for every automorphism φ of the relation R and every unit $X \in E$ it holds

$$t(X) = t(\varphi(X))$$

Then we have

$$X \equiv Y \Rightarrow t(X) = t(Y)$$

Examples of such properties are

$$t(U) = \text{ degree (number of neighbours) of unit } U$$

or (see [3])

$$t(U) = \text{ number of units at distance } d \text{ from the unit } U$$

or (see [11])

$$t(U) = \text{ number of triads of type } x \text{ at the unit } U$$

For other examples see [16].

The collection of structural properties $t_1, t_2, \ldots, t_m$ is *complete* (for structural equivalence) iff for each pair of units X and Y it satisfies the condition

$$(\forall i, 1 \leq i \leq m : t_i(X) = t_i(Y)) \Rightarrow X \equiv Y$$

Again, we can define the description of the unit U as

$$[U] = [t_1(U), t_2(U), \ldots, t_m(U)]$$

and the dissimilarity between units U and V as

$$d(U, V) = D([U], [V])$$

where D is some (standard) dissimilarity between real vectors.

In the case when the dissimilarity D has the property

$$D([U], [V]) = 0 \Rightarrow [U] = [V]$$

and the properties $t_1, t_2, \ldots, t_m$ are complete, it holds

$$d(X, Y) = 0 = D([X], [Y]) \Leftrightarrow [X] = [Y] \Leftrightarrow \forall i : t_i(X) = t_i(Y) \Leftrightarrow X \equiv Y$$

Therefore we finally have

$$d(X, Y) = 0 \Leftrightarrow X \equiv Y$$

REMARK: The triads are not complete. Counterexample: path of length 3. The extreme units have the same triadic spectrum, but they are not structurally equivalent. ■

2.3 Matrix dissimilarities

2.3.1 Dissimilarities

The following is a list of dissimilarities for measuring the similarity between units X_i and X_j:

Manhattan distance

$$d_m(X_i, X_j) = \sum_{s=1}^{n} (|q_{is} - q_{js}| + |q_{si} - q_{sj}|)$$

Euclidean distance

$$d_E(X_i, X_j) = \sqrt{\sum_{s=1}^{n} ((q_{is} - q_{js})^2 + (q_{si} - q_{sj})^2)}$$

Truncated Manhattan distance

$$d_s(X_i, X_j) = \sum_{\substack{s=1 \\ s \neq i,j}}^{n} (|q_{is} - q_{js}| + |q_{si} - q_{sj}|)$$

Truncated Euclidean distance [13]

$$d_S(X_i, X_j) = \sqrt{\sum_{\substack{s=1 \\ s\neq i,j}}^{n} ((q_{is} - q_{js})^2 + (q_{si} - q_{sj})^2)}$$

Corrected Manhattan-like dissimilarity ($p > 0$)

$$d_c(p)(X_i, X_j) = d_s(X_i, X_j) + p \cdot (|q_{ii} - q_{jj}| + |q_{ij} - q_{ji}|)$$

Corrected Euclidean-like dissimilarity [9]

$$d_e(p)(X_i, X_j) = \sqrt{d_s(X_i, X_j)^2 + p \cdot ((q_{ii} - q_{jj})^2 + (q_{ij} - q_{ji})^2)}$$

Corrected dissimilarity

$$d_C(p)(X_i, X_j) = \sqrt{d_c(p)(X_i, X_j)}$$

It is easy to verify that all expressions from the list define a dissimilarity, i.e., they have the properties

d1. $d(X, Y) \geq 0$
d2. $d(X, X) = 0$
d3. $d(X, Y) = d(Y, X)$

A dissimilarity d which has also the properties

d4. $d(X, Y) = 0 \Rightarrow X = Y$
d5. $d(X, Y) + d(Y, Z) \geq d(X, Z)$

is called a *distance*. Each of the dissimilarities from the list can be assessed to see whether or not it is also a distance.

REMARK: Because different units may have equal descriptions (rows and columns) the property

$$d(X_i, X_j) = 0 \Rightarrow X_i = X_j$$

does not hold. Therefore the listed dissimilarities can not be distances in a strict sense. Nevertheless we can require a slightly relaxed property

$$d(X_i, X_j) = 0 \Leftrightarrow X_i \sim X_j$$

■

REMARK: In the case $q = r$ all the dissimilarities from the list are invariant to the complementing of relation $R \to E \times E \setminus R = \bar{R}$, i.e., $d(R)(X, Y) = d(\bar{R})(X, Y)$. ■

2.3.2 Compatibility

In procedures based on structural equivalence we can expect the best results if we use a dissimilarity d which is *compatible* with structural equivalence, i.e.,

$$X_i \equiv X_j \Leftrightarrow d(X_i, X_j) = 0$$

Not all the dissimilarities from the list are compatible.

If q is sensitive, for dissimilarities d_m and d_E only

$$d(X_i, X_j) = 0 \Rightarrow X_i \equiv X_j$$

holds. The converse does not hold. For the matrix

$$\mathbf{R} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

the units X_1 and X_2 are structurally equivalent, but $d_m(X_1, X_2) = 2$ and $d_m(X_1, X_2) = \sqrt{2}$, for $q = r$.

If q is a structural property, for dissimilarities d_S and d_s only

$$X_i \equiv X_j \Rightarrow d(X_i, X_j) = 0$$

holds. The converse does not hold. For the matrix

$$\mathbf{R} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

for $q = r$, $d_S(X_1, X_2) = d_s(X_1, X_2) = 0$, but the units X_1 and X_2 are not structurally equivalent.

If structural property q is sensitive the dissimilarities d_c, d_C and d_e are compatible with the structural equivalence.

2.3.3 Triangle inequality

Which of the dissimilarities from the list satisfy the triangle inequality

$$d(X_i, X_k) + d(X_k, X_j) \geq d(X_i, X_j)$$

It holds for dissimilarities d_m and d_E.

We have to show that the quantity

$$\Delta = d(X_i, X_k) + d(X_k, X_j) - d(X_i, X_j)$$

is always nonnegative.

By complete enumeration of all possible cases for 0/1 matrices on the computer we found that there exist counterexamples for d_s. For example, for the matrix

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

we get

$$\Delta_s = |0-0| + |1-1| + |1-1| + |0-0| - |0-1| - |1-0| = -2$$

The same matrix is also a counterexample for d_S.

For the dissimilarity $d_c(p)$ there also exist counterexamples. For example, for the matrix

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

we get

$$\begin{aligned} \Delta_c(2) &= p \cdot (|0-0| + |0-0|) + |1-1| + |0-1| + \\ &+ p \cdot (|1-1| + |0-0|) + |1-0| + |0-0| - \\ &- p \cdot (|0-0| + |1-0|) - |0-1| - |0-1| = \\ &= 1 + 1 - (2+p) = -p \end{aligned}$$

Therefore the triangle inequality does not hold for all $p > 0$.

For $p = 2$ all the counterexamples have either the form $1 + 1 - 4 = -2$ or the form $1 + 3 - 6 = -2$. Since $\sqrt{1} + \sqrt{1} - \sqrt{4} = 0$ and $\sqrt{1} + \sqrt{3} - \sqrt{6} > 0$ the dissimilarity $\delta_C(2)$ satisfies the triangle inequality for 0/1 matrices.

For $p = 1$ all the counterexamples have either the form $1 + 1 - 3 = -1$ or the form $1 + 2 - 4 = -1$. Since $\sqrt{1} + \sqrt{1} - \sqrt{3} > 0$ and $\sqrt{1} + \sqrt{2} - \sqrt{4} > 0$ the dissimilarity $\delta_C(1)$ satisfies the triangle inequality for 0/1 matrices.

It is easy to show that also the dissimilarity $d_C(p), p = 1, 2$ satisfies the triangle inequality for 0/1 matrices.

Similary, by complete enumeration, we establish that the dissimilarity

$$\delta_e(p)(X_i, X_j) = \sqrt{p \cdot ((q_{ii} - q_{jj})^2 + (q_{ij} - q_{ji})^2) + (q_{ik} - q_{jk})^2 + (q_{ki} - q_{kj})^2}$$

satisfies the triangle inequality for 0/1 matrices for $p = 1, 2$. From this we can prove that d_e also satisfies triangle inequality for 0/1 matrices.

For real vectors the dissimilarity $\delta_e(2)$ does not satisfy the triangle inequality. For a counterexample let us consider the matrix

$$\begin{bmatrix} 5 & 9 & 6 \\ 0 & 7 & 4 \\ 4 & 6 & 6 \end{bmatrix}$$

We obtain

$$\Delta_e = \delta_e(X_i, X_k) + \delta_e(X_k, X_j) - \delta_e(X_i, X_j) = \sqrt{35} + \sqrt{35} - \sqrt{178} = -1.51$$

For the dissimilarities $\delta_e(1)$ and $\delta_C(1)$ over real vectors the answer is unknown. The random search for a counterexample with more than 1000000 trials failed.

For $\delta_C(2)$ over real numbers Martin Juvan and Marko Petkovšek [21] proved that the triangle inequality holds.

For $a, b \in \{0, 1\}$ the following equality

$$(a - b)^2 = |a - b|$$

holds. This implies some additional relations between dissimilarities from the list

$$\begin{aligned} d_E &= \sqrt{d_m} \\ d_S &= \sqrt{d_s} \\ d_e(p) &= d_C(p) \end{aligned}$$

It is interesting that for symmetrical matrices, $q_{uv} = q_{vu}$ for all u and v, we can prove that the dissimilarities d_s and d_c (and therefore also d_S, d_C and d_e) satisfy the triangle inequality.

3 Direct approach: optimization joint work with P. Doreian and A. Ferligoj

This section contains results from [5, 6].

The problem of determining the classes of equivalent units is called the *blocking problem*. It is very similar to the clustering problem.

Assume that we have a single relation network $\mathcal{N} = (E, R)$. Let Θ denote the set of all equivalence relations of a selected type (for example, regular or structural equivalences) over $\mathcal{N}$. Every equivalence relation $\sim$ on E determines a partition $\mathcal{C}$ of E, and vice versa.

Suppose that we are able to construct a criterion function $P(\mathcal{C})$ with the properties:

P1. $P(\mathcal{C}) \geq 0$
P2. $P(\mathcal{C}) = 0 \Leftrightarrow \sim \in \Theta$

Then we can express the blocking problem as an optimization problem:

Determine the clustering $\mathcal{C}^* \in \Phi$ such that

$$P(\mathcal{C}^*) = \min_{\mathcal{C} \in \Phi} P(\mathcal{C})$$

In the case when Θ is empty the optimization approach gives the solution(s) which differ(s) the less from some ideal case.

Given a clustering $\mathcal{C} = \{C_1, C_2, \ldots, C_k\}$, let $\mathcal{B}(C_u, C_v)$ denote the set of all ideal blocks corresponding to block $R(C_u, C_v)$. Then the global error of clustering $\mathcal{C}$ can be expressed as

$$P(\mathcal{C}) = \sum_{C_u, C_v \in \mathcal{C}} \min_{B \in \mathcal{B}(C_u, C_v)} d(R(Cu, C_v), B)$$

where the term $d(R(C_u, C_v), B)$ measures the difference (local error) between the block $R(C_u, C_v)$ and the ideal block B. The function d have to be compatible with the selected type of equivalence.

In the following we shall construct the function d for regular equivalence.

3.1 Criterion for structural equivalence

In paper [6] we proposed an approach to construction of criterion functions reflecting the notion of regular equivalence. The same approach can be used also in the case of structural equivalence.

From the definition of structural equivalence it follows that there are four possible ideal diagonal blocks $B(C, C)$

Type 1.	$b_{ij} = 0$
Type 2.	$b_{ij} = \delta_{ij}$
Type 3.	$b_{ij} = 1 - \delta_{ij}$
Type 4.	$b_{ij} = 1$

where δ_{ij} is the Kronecker delta function and $i, j \in C$.

For the nondiagonal blocks $R(C_u, C_v), u \neq v$ only blocks of type 1 and type 4 are possible.

Given a clustering $\mathcal{C} = \{C_1, C_2, \ldots, C_k\}$, let $\mathcal{B}(C_u, C_v)$ denote the set of all ideal blocks corresponding to block $R(C_u, C_v)$. Then the global error of clustering $\mathcal{C}$ can be expressed as

$$P(\mathcal{C}) = \sum_{C_u, C_v \in \mathcal{C}} p(C_u, C_v)$$

and

$$p(C_u, C_v) = \min_{B \in \mathcal{B}(C_u, C_v)} d(R(C_u, C_v), B)$$

where the obvious choice for d is

$$d(R(C_u, C_v), B) = \sum_{X \in C_u, Y \in C_v} |r_{xy} - b_{xy}|$$

It is easy to verify that so defined criterion function $P(\mathcal{C})$ is sensitive to structural equivalence

$$P(\mathcal{C}) = 0 \Leftrightarrow \mathcal{C} \text{ defines structural equivalence}$$

Beside this, it is invariant to the transformation of complementing the relation $R \rightarrow E \times E \setminus R$.

The selected dissimilarity between blocks is based on the assumption that the error $0 \rightarrow 1$ is equiprobable to the error $1 \rightarrow 0$. In the case that this assumption is not valid we can introduce the weights α and β which balance both types of errors. Let B^* be a solution of optimization problem from the right side of the expression for $p(C_u, C_v)$. Then we can redefine p, for example, as follows

$$\begin{aligned} p(C_u, C_v) &= \alpha \operatorname{card}\{(X, Y) \in C_u \times C_v : r_{xy} > b^*_{xy}\} \\ &+ \beta \operatorname{card}\{(X, Y) \in C_u \times C_v : r_{xy} < b^*_{xy}\} \end{aligned}$$

Selecting large α ($\alpha = 100$) and small β ($\beta = 1$), optimizing $P(\mathcal{C})$ we are primarily seeking solutions with the least number of errors of the type $0 \rightarrow 1$ and, as a secondary criterion, among them solutions with the least number of errors of the type $1 \rightarrow 0$.

In the special, but in applications surprisingly frequent, case when the number of 0's equals the number of 1's in the block $R(C_u, C_v)$, we decided to select 1's block for B^* if $\alpha \geq \beta$; and 0's block otherwise.

4 Criterion for regular equivalence

The construction of the criterion function for regular equivalence we shall base on the following observation.

PROPOSITION 4.1 *Let $\mathcal{C} = \{C_i\}$ be a clustering corresponding to a regular equivalence $\cong$ on the network $\mathcal{N} = (E, R)$. Then each block $R(C_u, C_v)$ is either null or it has the property that there is at least 1 in each of its rows and in each of its columns.*

Conversely, if for a given clustering $\mathcal{C}$ each block has this property then the corresponding equivalence relation is a regular equivalence.

From the proposition it follows that regular equivalence produces two types of blocks:

- *null blocks* which have all entries 0; and
- *1-covered blocks*, which have in each row and in each column at least one 1.

Therefore we can use as a measure of regularity of a block the quantity

$$d(R(C_u, C_v), B) = \begin{cases} \#\text{ of 1-covered rows/columns} & \text{B is null block} \\ \#\text{ of 0 rows/columns} & \text{B is 1-covered block} \end{cases}$$

From the proposition it follows that so defined criterion function $P(\mathcal{C})$ is sensitive to regular equivalence

$$P(\mathcal{C}) = 0 \Leftrightarrow \mathcal{C} \text{ defines regular equivalence}$$

If necessary we can introduce in d the weights to tune the importance of $1 \rightarrow 0$ and $0 \rightarrow 1$ errors.

4.1 Establishing block models

For solving the block modeling problem we use a local optimization algorithm:

Determine the initial clustering $\mathcal{C}$;
repeat:
  if in the neighborhood of the current clustering $\mathcal{C}$
  there exists a clustering $\mathcal{C}'$ such that $P(\mathcal{C}') < P(\mathcal{C})$
  then move to clustering $\mathcal{C}'$.

In the algorithm the *neighborhood* of a given clustering is defined by the following clustering transformations:

- *moving* a unit from one cluster to another cluster;
- *interchanging* of two units from different clusters.

We repeat the local optimization search with several random initial clusterings and preserve the best 10 obtained solutions.

In tables 1 and 2 the (locally) best solutions for Sampson liking data for the criterion based on structural equivalence are given.

For determining regular equivalencies in large data sets the *dynamic clusters method* from cluster analysis can be adapted. It was introduced and elaborated by E. Diday and his collaborators [12]. For two decades it has been used very successfully to solve different clustering problems, especially for large data sets.

The proposed criterion function to measure departures of a given clustering from exact regular partition [6] combined with the dynamic clusters method to search for regular clusterings of a given network [4].

		1	13	14	15	16	2	7	12	4	5	9	6	8	10	11	3	17	18
John Bosco	1	0	0	•	0	0	⋆	⋆	1	0	0	0	0	0	0	0	•	0	0
Amand	13	0	0	0	0	0	⋆	1	⋆	0	•	0	0	0	0	0	0	0	•
Hugh	14	•	0	0	•	0	⋆	⋆	1	0	0	0	0	0	0	0	0	0	0
Boniface	15	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0
Albert	16	0	0	0	•	0	1	1	⋆	0	0	0	0	0	0	0	0	0	0
Gregory	2	•	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0
Mark	7	0	0	0	0	•	1	0	1	0	0	0	0	0	0	0	0	0	0
Winfrid	12	•	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0
Peter	4	0	0	0	0	0	0	0	0	0	1	⋆	•	0	0	•	0	0	0
Bonaventur	5	0	0	0	0	0	0	0	0	1	0	1	0	0	0	•	0	0	0
Ambrose	9	0	0	0	0	0	0	0	•	⋆	1	0	0	•	0	0	0	0	0
Berthold	6	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0
Victor	8	0	0	0	0	0	0	0	0	1	⋆	1	•	0	0	0	0	0	0
Romuald	10	0	•	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0
Louis	11	0	0	•	0	0	0	0	0	⋆	1	⋆	0	•	0	0	0	0	0
Basil	3	•	•	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1
Elias	17	0	0	0	0	0	•	0	0	0	0	0	0	0	0	0	1	0	1
Simplicius	18	0	0	0	0	0	•	0	0	0	0	0	0	0	0	0	1	1	0

Table 1: Sampson Data - Liking / LocOpt $k = 5, \alpha = 1, \beta = 1$

4.2 Multicriteria approach to block models

Multicriteria clustering algorithms [15] provide an alternative approach for obtaining structural and regular equivalences in multi-relational networks. Appropriate dissimmilarity measures between units and criterion functions are defined [14].

5 Other topics

5.1 Metric degree of dissimilarity

Let $d(X, Y)$ be a dissimilarity over E. Then

$$d_r(X, Y) = (d(X, Y))^r, \qquad r > 0$$

is also a dissimilarity. Evidently d_0 satisfies the triangle inequality. Therefore we can define the *metric degree* of dissimilarity d as the greatest $R \in \mathbb{R}$ such that for all $r, 0 < r \leq R$ the dissimilarity d_r satisfies the triangle inequality.

5.2 Two additional network semirings

At the last seminar meeting [2] I presented the application of semirings to balancing and clusterability problem for signed graphs.

Studying the network literature I noticed that the semiring

$$(\{0, 1, 2, 3\}, \max, \min)$$

can be applied to determine the connectedness matrix [18, 133].

Another example of a semiring I found reading the book [17, 34-38,111-112]. For computing the Freeman's indices we need the number of u-v geodesics for each pair of

		1	15	2	7	12	14	16	4	5	6	8	9	11	10	3	13	17	18
John Bosco	1	0	0	⋆	⋆	1	1	⋆	0	0	0	0	0	0	0	•	0	0	0
Boniface	15	0	0	1	1	1	⋆	⋆	0	0	0	0	0	0	0	0	0	0	0
Gregory	2	1	⋆	0	1	1	⋆	⋆	0	0	0	0	0	0	0	0	0	0	0
Mark	7	⋆	⋆	1	0	1	⋆	1	0	0	0	0	0	0	0	0	0	0	0
Winfrid	12	1	⋆	1	1	0	⋆	⋆	0	0	0	0	0	0	0	0	0	0	0
Hugh	14	1	1	⋆	⋆	1	0	⋆	0	0	0	0	0	0	0	0	0	0	0
Albert	16	⋆	1	1	1	⋆	⋆	0	0	0	0	0	0	0	0	0	0	0	0
Peter	4	0	0	0	0	0	0	0	0	1	1	⋆	⋆	1	0	0	0	0	0
Bonaventur	5	0	0	0	0	0	0	0	1	0	⋆	⋆	1	1	0	0	0	0	0
Berthold	6	0	0	0	0	0	0	0	1	1	0	⋆	1	⋆	0	0	0	0	0
Victor	8	0	0	0	0	0	0	0	1	⋆	1	0	1	⋆	0	0	0	0	0
Ambrose	9	0	0	0	0	•	0	0	⋆	1	⋆	1	0	⋆	0	0	0	0	0
Louis	11	0	0	0	0	0	•	0	⋆	1	⋆	1	⋆	0	0	0	0	0	0
Romuald	10	0	0	0	0	0	0	0	1	1	⋆	⋆	1	⋆	0	0	•	0	0
Basil	3	•	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1
Amand	13	0	0	0	•	0	0	0	0	•	0	0	0	0	0	⋆	0	⋆	1
Elias	17	0	0	•	0	0	0	0	0	0	0	0	0	0	0	1	⋆	0	1
Simplicius	18	0	0	•	0	0	0	0	0	0	0	0	0	0	0	1	⋆	1	0

Table 2: Sampson data – liking / LocOpt $k = 5, \alpha = 100, \beta = 1, P = 941$

vertices (u, v). This numbers can be obtained by computing the closure of relation matrix over the following *geodetic semiring*:

First we transform relation R to a matrix which has for entries pairs defined by

$$(d, n)_{u,v} = \begin{cases} (1, 1) & (u, v) \in R \\ (\infty, 0) & (u, v) \notin R \end{cases}$$

where d is the length of the shortest path and n is the number of shortest paths.

In the set $A = \bar{\mathbb{R}}_0^+ \times \mathbb{N}$ we define the operations

$$(a, i) \oplus (b, j) = (\min(a, b), \begin{cases} i & a < b \\ i + j & a = b \\ j & a > b \end{cases})$$

$$(a, i) \odot (b, j) = (a + b, i.j)$$

It is not difficult to verify that $(A, \oplus, \odot)$ is indeed a semiring with zero $(\infty, 0)$ and identity $(0, 1)$. It is also closed, with a closure

$$(a, i)^\star = \begin{cases} (0, \infty) & a = 0, i \neq 0 \\ (0, 1) & \text{otherwise} \end{cases}$$

5.3 Citation networks

Reading Norm's and Pat's papers on citation networks [19, 20] I noticed that the arc weights (number of different paths which contains the arc) can be more efficiently computed by first computing for each vertex v the number $N^+(v)$ of ways it can be reached from initial vertices and the number $N^-(v)$ of ways it can reach the terminal vertices of network. The weight $w(u, v)$ of the arc (u, v) is then

$$w(u, v) = N^+(u) \cdot N^-(v)$$

The program in pascal implementing this idea is given in the appendix. It assumes that the vertices of the network are topologically sorted.

Because for each internal vertex in the network the number of in-going paths equals the number of out-going paths for the weights w the *Kirchoff's node law* holds. This can be seen also formally

$$\sum_{(t,v)\in R} w(t,v) = \sum_{(t,v)\in R} N^+(t)\cdot N^-(v) = (\sum_{(t,v)\in R} N^+(t))\cdot N^-(v) = N^+(v)\cdot N^-(v)$$

and

$$\sum_{(v,u)\in R} w(v,u) = \sum_{(v,u)\in R} N^+(v)\cdot N^-(u) = N^+(v)\cdot \sum_{(v,u)\in R} N^-(u) = N^+(v)\cdot N^-(v)$$

The quantity $N^+(v)\cdot N^-(v)$ equals to the number of all paths through the vertex v.

By connecting all initial vertices to a single source s and all terminal vertices to a single sink t and adding the arc (t,s) we can extend the Kirchoff's law to the entire network.

The max-flow (number of all paths from initial to terminal vertices) can be used as an appropriate normalization factor.

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Path Count Procedure

```
PROCEDURE count(VAR a, b: entries; VAR c: counters);
VAR
   norm : Boolean;
   v, u, t : vertex;
   p, q : p_arc;
   w : LongInt;
BEGIN
   norm := (NOT source) AND forwrd;
   FOR v := 0 TO NumUnitsp DO b[v] := NIL;
   v := start; t := 0;
   IF norm THEN c[v] := 0 ELSE c[v] := 1;
   REPEAT
      p := a[v];
      IF norm AND (v > 0) THEN c[v] := c[v] + 1;
      w := c[v];
      WHILE p <> NIL DO BEGIN
         u := p^.vtx; q := p; p := p^.nxt;
         c[u] := c[u] + w;
      { transfer current arc (v,u) to inverse relation }
         q^.vtx := v; q^.nxt := b[u]; b[u] := q;
      END;
      IF c[v] = 0 THEN c[v] := 1;
      u := order[v]; order[v] := t; t := v; v := u;
   UNTIL v = 0;
   start := t;
END {count};

PROCEDURE counter;
VAR
   bfirst    { backward successors lists entries }
          : entries;
BEGIN
   FOR v := 0 TO NumUnitsp DO BEGIN fcount[v] := 0; bcount[v] := 0 END;
   writeln(lst); writeln(lst,'forward  count'); forwrd := TRUE;
   count(ffirst,bfirst,fcount);
   writeln(lst); writeln(lst,'backward count'); forwrd := FALSE;
   count(bfirst,ffirst,bcount);
END {counter};
```